

Transient dynamics of open quantum systems

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(Dated: May 23, 2013)

We present a renormalization group (RG) method which allows for an analytical study of the transient dynamics of open quantum systems on all time scales. Whereas oscillation frequencies and decay rates of exponential time evolution follow from the fixed point positions, the long-time behavior of pre-exponential functions is related to the scaling behavior around the fixed points. We show that certain terms of the RG flow are only cut off by inverse time, which leads to a difference between infrared and ultraviolet scaling. An evaluation for the Ohmic spin boson model at weak damping reveals significant deviations from previous predictions in the long-time regime. We propose that weak-coupling problems for stationary quantities can in principle turn into strong-coupling ones for the determination of the long-time behavior.

PACS numbers: 05.10.Cc, 05.30.-d, 05.30.Jp, 73.23.-b

The time dynamics of a small strongly interacting quantum system coupled to noninteracting large reservoirs is a fundamental issue in nonequilibrium statistical mechanics. The prototype is a two-level system coupled to an environment,^{1,2} which is of particular interest in quantum information processing.³ A typical setup is the one of transient dynamics: The system and environment are decoupled for times $t < 0$ and the coupling is switched on suddenly at $t = 0$. The time evolution of the reduced density matrix ρ_t of the local system will then be characterized for $t > 0$ by a series of terms, each of which will generically be of the form of an exponential together with a pre-exponential function $\rho_t = \sum_n F_t^n \exp(-iz_n t)$. Here, $z_n = \Omega_n - i\Gamma_n$ consists of an oscillation frequency Ω_n and a decay rate Γ_n , where one of the scales z_n will be zero characterizing the stationary state ρ_{st} for $t \rightarrow \infty$. Besides the calculation of ρ_{st} and z_n , the main challenge lies in the analysis of the pre-exponential functions F_t^n on all time scales. Although interesting field-theoretical⁴ and numerical techniques⁵ have been developed to study the time dynamics, the precise form of pre-exponential functions has not been addressed so far. Promising tools for this purpose are perturbative renormalization group (RG) methods for nonequilibrium problems, like the flow equation method,^{6,7} real-time RG (RTRG)^{8–11} and functional RG¹² techniques, or combinations of the latter two.¹³ The RTRG method allows for an analytical study on all time scales, provided that the RG flow stays in the weak-coupling regime. The time dynamics is related to the density matrix $\rho(E)$ in Laplace space, where the exponential scales z_n are the singularities of $\rho(E)$ in the complex plane and the pre-exponential functions can be determined from branch cut integrals starting at these singularities. In Ref. 11 a RG approach has been proposed by using the Laplace variable E itself as flow parameter (called E-RTRG in the following), where the singularities z_n are given by the fixed points of the RG flow and the long-time behavior of pre-exponential functions can be related to the scaling behavior around the fixed points.

In this Rapid Communication we will combine E-RTRG with a new parametrization of the effective Liouvillian in terms of slowly varying logarithmic functions and provide a discussion of the generic time evolution on all time scales. The main result is the insight that, for the determination of pre-exponential functions, certain terms of the RG flow are only cut off by the energy scale of inverse time $1/t$. This is in contrast to stationary quantities, where it has been proposed^{14,15} and microscopically shown^{16,17} that *all* terms of the RG flow are cut off by decay rates. As a consequence, we find that the long-time behavior is generically quite different from that discussed in Refs. 9 and 10 at intermediate and short times. To show this explicitly we will apply our method to the Ohmic spin boson model at weak damping, which turns out to be a weak-coupling problem even close to the fixed points. For the diagonal components of the density matrix, we find that the power-law exponent for the scaling behavior of the pre-exponential function agrees with perturbation theory, in contrast with that predicted by the noninteracting blip approximation (NIBA).^{1,2} For the nondiagonal elements we find a rather complex scaling behavior which differs from that of perturbation theory. We expect similar deviations to occur for other models of open quantum systems as well. In particular, for certain problems, e.g., the antiferromagnetic nonequilibrium Kondo model at large bias voltage, it may even turn out that the renormalized vertices are small for the calculation of stationary quantities, but become large close to the fixed points z_n , i.e., a weak-coupling problem for stationary quantities can turn into a strong-coupling one for the study of the long-time behavior. This opens up another class of interesting problems for the future.

Generic discussion.— For a discrete quantum system coupled to noninteracting reservoirs at time $t = 0$, it can be shown¹⁸ that the time evolution of the reduced density matrix ρ_t of the local system follows from $i\dot{\rho}_t = \int_0^t dt' L_{t-t'} \rho_{t'}$, where L_t is an effective Liouvillian superoperator acting on local operators. Defining the Laplace transform via $A(E) = \int_0^\infty dt e^{iEt} A_t$, one obtains

the formal solution $\rho(E) = i\Pi(E)\rho_{t=0}$ with the propagator $\Pi(E) = \frac{1}{E-L(E)}$ and the time evolution follows from the inverse Laplace transform

$$\rho_t = \frac{i}{2\pi} \int_{-\infty+i0^+}^{\infty+i0^+} dE e^{-iEt} \Pi(E) \rho_{t=0} \quad (1)$$

Reference 8 shows how to calculate the Liouvillian $L(E)$ from a diagrammatic expansion in some appropriately defined dimensionless system-bath coupling α . The propagator $\Pi(E)$ is an analytic function in the upper half of the complex plane with poles at z_k^p ($k = \text{st}, 0, \pm 1, \pm 2, \dots$) in the lower half of the complex plane. $k = \text{st}$ denotes the zero pole $z_{\text{st}}^p = 0$, which defines the stationary state. At zero temperature,¹⁹ additional nonanalytic features arise from branch cuts starting at the singularities $z_n = z_k^p + \Delta\mu_n$, which are generically given by the pole positions shifted by some linear combination $\Delta\mu_n$ of the chemical potentials of the reservoirs. They arise since the Liouvillian depends logarithmically on terms $\sim \alpha \ln(\frac{D}{E-z_n})$, generated by ultraviolet divergencies in the band width D of the reservoirs. The logarithmic divergencies can be systematically resummed by using E-RTRG.¹¹ The RG equations express derivatives of the Liouvillian $L(E)$ by a diagrammatic series in terms of effective vertices, which is free of divergences and can be systematically truncated for weak-coupling problems. Solving the RG equations along the path $E = z_n + i\Lambda \pm 0^+$ in the complex plane starting at $\Lambda = D$, one can tune the positions of the branch cuts to $E = z_n - ix$, $x > 0$.

For a discussion of the generic time evolution, it is very helpful to use the form

$$L(E) = L_\Delta(E) + E L'(E) \quad , \quad (2)$$

where $L_\Delta(E)$ and $L'(E)$ are slowly varying logarithmic functions. This form is valid in the universal regime $|E| \ll D$. The nonuniversal regime $|E| \gtrsim D$, which corresponds to the *ultrashort time regime* $t \lesssim 1/D$, is not of interest here since it depends on the microscopic details of the high-energy cutoff function. In the supplementary material²⁰ it is shown how the E-RTRG method can be used to obtain RG equations for $\tilde{L}_\Delta(E) = Z'(E)L_\Delta(E)$ and the Z -factor superoperator $Z'(E) = 1/[1 - L'(E)]$. With these quantities one can express the time evolution (1) in a form where the slowly varying logarithmic parts are explicitly shown

$$\rho_t = \frac{i}{2\pi} \int_{-\infty+i0^+}^{\infty+i0^+} dE e^{-iEt} \tilde{\Pi}(E) Z'(E) \rho_{t=0} \quad (3)$$

$$= \sum_k \frac{i}{2\pi} \int_\gamma dE \frac{e^{-iEt}}{E - \lambda_k(E)} P_k(E) Z'(E) \rho_{t=0} \quad , \quad (4)$$

where $\tilde{\Pi}(E) = \frac{1}{E - \tilde{L}_\Delta(E)}$. $\lambda_k(E)$ are the eigenvalues of $\tilde{L}_\Delta(E)$ with projectors $P_k(E)$. The poles z_k^p follow from $z_k^p = \lambda_k(z_k^p)$. The integral is performed by closing the integration contour γ in the lower half of the complex

plane. Using the general expressions (1), (3) and (4), the typical time dynamics can be obtained as follows:

For *short times*, $t \ll 1/|z_n|$, one can replace $E \rightarrow 1/t$ in the logarithmic functions \tilde{L}_Δ and Z' . This gives for Eq. (3) the result $\rho_t \approx e^{-i\tilde{L}_\Delta(1/t)t} Z'(1/t) \rho_{t=0}$. Expanding the exponential, one finds in leading order that the scaling behavior at small times follows from the scaling behavior of $Z'(1/t)$ at large energies. This is the poor man scaling regime, where the cutoff scales z_n are unimportant. As a result, one obtains universal short-time behavior, which has been reported, e.g., for spin boson^{1,2} and Kondo models^{7,9}.

For *intermediate and long times*, $t \gtrsim 1/|z_n|$, we consider each integral separately around the branch cut at $E = z_n - ix$, $x > 0$. It leads to an exponential factor $e^{-iz_n t}$ multiplied by the pre-exponential function $F_t^n = \frac{1}{2\pi} \int_0^\infty e^{-xt} \delta f(z_n - ix)$, where $\delta f(E) = f(E+0^+) - f(E-0^-)$ denotes the jump of the integrand across the branch cut. We start with the contribution from a *branching pole* $z_n \equiv z_k^p$. Here, one can approximately replace $x \rightarrow 1/t$ in all logarithmic functions and take the average value $\bar{P}_k(z_k^p - i/t)$ and $\bar{Z}'(z_k^p - i/t)$ across the branch cut. The jump over the branch cut is dominated by a delta function from the resolvent $\delta_{\frac{1}{E - \lambda_k(E)}} \approx 2\pi\delta(x)$. This gives the contribution $\rho_t^{k,p}$ from the branching pole z_k^p

$$\rho_t^{k,p} \approx e^{-iz_k^p t} \bar{P}_k(z_k^p - \frac{i}{t}) \bar{Z}'(z_k^p - \frac{i}{t}) \rho_{t=0} \quad (5)$$

We obtain a logarithmic scaling of the pre-exponential function $F_t^{k,p}$, which follows from the scaling behavior of $\bar{P}_k(E)$ and $\bar{Z}'(E)$ around the fixed point z_k^p . For *intermediate times*, where one can expand the logarithmic scaling perturbatively in $\alpha \ln(|z_n|t) \ll 1$, one obtains the weak-coupling expansion of Ref. 9. However, for *long times*, the correct scaling behavior has to be determined from a systematic expansion of the full solution of the RG equations around the fixed points. For $k = \text{st}$ and $t \rightarrow \infty$ one obtains the stationary distribution $\rho_{\text{st}} = |\bar{x}_{\text{st}}(0)\rangle$ from Eq. (5), with $\tilde{L}_\Delta(0)|x_{\text{st}}(0)\rangle = 0$.^{20,21}

For a *branching point* z_n , it is more convenient to start from Eq. (1) and write for the jump across the branch cut $\delta\Pi = \Pi_+(\delta L)\Pi_-$ with $\Pi_\pm = \Pi(z_n - ix \pm 0^+)$. Using $\Pi_\pm = \tilde{\Pi}_\pm Z'_\pm$, neglecting terms of $O(\delta L)^3$, and approximating $x \rightarrow 1/t$ in the logarithmic functions, we obtain after some straightforward manipulations the contribution $\rho_t^{n,b}$ from the branching point z_n :

$$\rho_t^{n,b} \approx e^{-iz_n t} \sum_{k,k' \neq n} \frac{1}{2\pi} \int_0^\infty dx e^{-xt} \times \bar{P}_k^n \bar{Z}'^n \frac{\delta L(z_n - ix)}{(z_n - ix - \bar{\lambda}_k^n)(z_n - ix - \bar{\lambda}_{k'}^n)} \bar{P}_{k'}^n \bar{Z}'^n \rho_{t=0} \quad , \quad (6)$$

where $\bar{A}^n = \bar{A}(z_n - i/t)$. For times $t \gg 1/|z_n - z_{k,k'}^p|$, the argument $x \sim 1/t$ in the denominators can usually be neglected.²² In this case and for *intermediate times*, where logarithmic scaling is unimportant, we obtain the

power law $F_t^{n,b} \sim 1/t^{1+r}$, where the exponent r follows from the scaling of the jump of the Liouvillian across the branch cut $\delta L(z_n - ix) \sim x^r$. We therefore set up an equation for $\frac{d\delta L}{dE}(z_n - ix)$ and solve this RG equation for $x \rightarrow 0$. For example, we find $r = 0$ for quantum dots in the charge fluctuation regime and $r = 1$ for the Kondo model and the Ohmic spin boson model.²⁰ For *long times*, one has to consider in addition the logarithmic corrections from the RG flow close to the fixed points.

Ohmic spin boson model.— We now apply our flexible method to the Ohmic spin boson model at zero bias where a bosonic reservoir $H_{\text{res}} = \sum_q \omega_q a_q^\dagger a_q$ is coupled to a two-level system with tunneling Δ , described by the Hamiltonian $H = -\frac{\Delta}{2}\sigma_x$. The coupling is given by $V = \frac{1}{2}\sigma_z \sum_q g_q(a_q + a_q^\dagger)$ with ohmic spectral density $J(\omega) = \pi \sum_q g_q^2 \delta(\omega - \omega_q) = 2\pi\alpha\omega\theta(\omega)\rho(\omega)$, where $\rho(\omega) = D^2/(D^2 + \omega^2)$ is some high-energy cutoff function. We first summarize our results and compare them to previous works. For weak damping $\alpha \ll 1$, we find three nonzero poles at $z_0 = -i\Gamma$ and $z_\pm = \pm\Omega - i\Gamma/2$, with the effective tunneling $\Omega = \Delta(\frac{\Omega}{D})^\alpha$ and $\Gamma = \pi\alpha\Omega$. In leading order truncation it turns out that no branching poles appear. Therefore, according to the general expressions (5) and (6), our results in the intermediate and long time regime $t \gg 1/\Omega$ can be written as

$$\rho_t = \rho_{\text{st}} + \sum_{n=0,\pm} (F_t^{n,p} + F_t^{n,b})e^{-iz_n t}, \quad (7)$$

with

$$(\rho_{\text{st}})_{\sigma\sigma} = \frac{1}{2}, \quad (\rho_{\text{st}})_{\sigma,-\sigma} = \frac{\Omega}{2\Delta} \quad (8)$$

$$F_t^{0,p} = \frac{\Omega}{\Delta} \begin{pmatrix} 0 & 0 \\ -1 & \Omega/\Delta \end{pmatrix} \otimes \tau_+ \quad \rho_{t=0}, \quad (9)$$

$$F_t^{\sigma,p} = \frac{1}{2} \begin{pmatrix} 1 & \sigma\Omega/\Delta \\ \sigma\Omega/\Delta & (\Omega/\Delta)^2 \end{pmatrix} \otimes \tau_- \quad \rho_{t=0}, \quad (10)$$

$$F_t^{0,b} = -\frac{2\alpha}{(\Omega t)^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \tau_- \quad \rho_{t=0}, \quad (11)$$

$$F_t^{\sigma,b} = -\frac{\alpha f_t}{(\Delta t)^2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \tau_+ \quad \rho_{t=0}, \quad (12)$$

where $\sigma = \pm$, $\tau_\pm = \frac{1}{2}(1 \pm \sigma_x)$, and

$$f_t = \{[1 + \alpha \ln(\Omega t)][1 - \ln(1 + \alpha \ln(\Omega t))]\}^{-2}. \quad (13)$$

In Liouville space we ordered the four possible states as $++$, $--$, $+-$, $-+$, where \pm denote the two local spin states. For two 2×2 -matrices A and B , we defined the 4×4 -matrix

$$A \otimes B \equiv \begin{pmatrix} A_{11}B & A_{12}B \\ A_{21}B & A_{22}B \end{pmatrix}.$$

The results (7)–(12) can also be obtained from Born²³ and the self-consistent Born²⁴ approximation, but the unrenormalized tunneling $\Omega \rightarrow \Delta$ appears and the pre-exponential functions can only be calculated up to $O(\alpha)$,

i.e., the logarithmic function f_t in Eq. (12) is missing.²⁵ This correction can only be obtained from a resummation of all leading logarithmic divergences at low energies. This poses the question of why a similar logarithmic correction is not obtained in Eq. (11). If one compares our result to the NIBA approximation,^{1,2} which discusses only the time dynamics of $\langle \sigma_z \rangle_t$ (for $\langle \sigma_{x,y} \rangle_{t=0} = 0$ and $\langle \sigma_z \rangle_{t=0} = 1$) and reads

$$\langle \sigma_z \rangle_t = e^{-\frac{\Gamma}{2}t} \cos(\Omega t) - \frac{2\alpha}{(\Omega t)^{2-2\alpha}}, \quad (14)$$

one finds, besides the missing exponential part $e^{-\Gamma t}$ in the second term,²⁶ a different power law exponent than the one predicted by our result (11). Below we will show that there is a subtle reason why all leading logarithmic divergencies cancel out in Eq. (11), which is due to the fact that the scaling of the vertex and the Z -factors is completely different around the fixed points compared with the scaling at high energies. Our results are further substantiated by bare perturbation theory up to $O(\alpha^2)$ ²⁷, which confirms that there are no logarithmic terms $\sim \alpha^2 \ln(\Omega t)$ in the time dynamics, consistent with Eqs. (11)–(13) [note that $f_t = 1 + O(\alpha^2 \ln^2(\Omega t))$].

We now sketch the derivation of our results. First we note that the full effective Liouvillian is decomposed as $L = L_\Delta(E) + EL'(E) + L^s$, where

$$L^s = i\pi\alpha\Delta \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \tau_+$$

is the part which arises from integrating out the symmetric part of the reservoir contractions. The other parts can be parametrized in the following way:

$$\tilde{L}_\Delta = \sum_\sigma \begin{pmatrix} 0 & 0 \\ 0 & -i\Gamma_\sigma \end{pmatrix} \otimes \tau_\sigma + \begin{pmatrix} 0 & \Delta \\ Z_- \Delta & 0 \end{pmatrix} \otimes \tau_-,$$

$$Z' = \sum_\sigma \begin{pmatrix} 1 & 0 \\ 0 & Z_\sigma \end{pmatrix} \otimes \tau_\sigma.$$

As a consequence, the full propagator can be written as $\Pi = \tilde{\Pi} Z' (1 + L^s/E)$, with $\tilde{\Pi} = 1/(E - \tilde{L}_\Delta)$. In leading order, $\Gamma_- \lesssim \alpha^2 \sqrt{Z_-} \Delta$ can be neglected and the correction from L^s influences only the stationary state (8) and the pole contribution (9). The Liouvillian \tilde{L}_Δ has four eigenvalues $\lambda_{\text{st}} = 0$, $\lambda_0 = -i\Gamma_+$ and, neglecting Γ_- , $\lambda_\pm = \pm\sqrt{Z_-} \Delta$, with corresponding projectors given by

$$P_{\text{st}} Z' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \tau_+,$$

$$P_0 Z' = \begin{pmatrix} 0 & 0 \\ 0 & Z_+ \end{pmatrix} \otimes \tau_+,$$

$$P_\pm Z' = \frac{1}{2} \begin{pmatrix} 1 & \pm\sqrt{Z_-} \\ \pm\sqrt{Z_-} & Z_- \end{pmatrix} \otimes \tau_-.$$

The jump of the Liouvillian across the branch cuts needed for the evaluation of (6) is parametrized as

$$\delta L = \sum_\sigma \begin{pmatrix} 0 & 0 \\ 0 & -i\delta\gamma_\sigma \end{pmatrix} \otimes \tau_\sigma.$$

In leading order, we obtain the following RG equations for Z_σ , Γ_+ and $\delta\gamma_\sigma$:²⁰

$$\frac{dZ_+}{dE} = \alpha\tilde{g} \sum_\sigma \frac{Z_+}{E - \lambda_\sigma}, \quad \frac{dZ_-}{dE} = \alpha\tilde{g} \frac{2Z_-}{E - \lambda_0} \quad (15)$$

$$\frac{d\Gamma_+}{dE} = i\alpha\tilde{g} \sum_\sigma \frac{\lambda_0 - \lambda_\sigma}{E - \lambda_\sigma} \quad (16)$$

$$\frac{d\delta\gamma_+}{dE} = 2\pi\theta(x)\alpha g^2 Z_- \quad \text{for } E = z_\sigma - ix \quad (17)$$

$$\frac{d\delta\gamma_-}{dE} = 4\pi\theta(x)\alpha g^2 Z_+ \quad \text{for } E = z_0 - ix \quad (18)$$

$$\frac{dg}{dE} = \alpha\tilde{g} \sum_\sigma \frac{1}{\lambda_\sigma} \ln(-i(E - \lambda_\sigma)) \quad (19)$$

The RG equations are coupled to the renormalization of the vertex function $\tilde{g} = Z_+ Z_- g^2$. The initial conditions at $E = iD$ are $Z_\sigma = 1$, $\Gamma_+ = \delta\gamma_\pm = 0$, and $g^2 = 1$. The solution of these RG equations is very different for high and low energies. For energies $|E| \gg \Omega$ we find $Z_\pm \approx (\frac{-iE}{D})^{2\alpha}$, $\tilde{g} \approx 1$, and $\Gamma_+ \approx 0$. According to our general analysis, this gives rise to the universal shorttime behavior

$$\rho_t = \begin{pmatrix} 1 & 0 \\ 0 & (\frac{1}{Dt})^{2\alpha} \end{pmatrix} \otimes \mathbb{1} \rho_{t=0},$$

which agrees with previous predictions.^{1,2} In contrast, around the fixed points we find a different scaling. Close to $z_0 = -i\Gamma$, we get $Z_+ \approx (\frac{\Omega}{\Delta})^2$, $Z_- \approx (\frac{\Omega}{\Delta})^2(1 - 2\alpha \ln \frac{-i(E-z_0)}{\Omega})^{-1}$, $\Gamma_+ \approx \Gamma$, $g^2 \approx (\frac{\Delta}{\Omega})^4$, and $\delta\gamma_-(z_0 - ix) \approx -i4\pi\alpha(\frac{\Delta}{\Omega})^2 x\theta(x)$. Close to $z_\sigma = \sigma\Omega - i\Gamma/2$, we obtain $Z_+ \approx (\frac{\Omega}{\Delta})^2(1 - \alpha \ln \frac{-i(E-z_\sigma)}{\Omega})^{-1}$, $Z_- \approx (\frac{\Omega}{\Delta})^2$, $\Gamma_+ \approx \Gamma/2 + i\sigma\Omega \ln(1 - \alpha \ln \frac{-i(E-z_\sigma)}{\Omega})$, $g^2 \approx (\frac{\Delta}{\Omega})^4$, and $\delta\gamma_+(z_\sigma - ix) \approx -i2\pi\alpha(\frac{\Delta}{\Omega})^2 x\theta(x)$. Inserting these results in the general expressions (5) and (6), one obtains the results (7-13) using straightforward algebra. As we can see, the most important point is that the vertex function $\tilde{g} = Z_+ Z_- g^2$ is approximately a constant for high energies, whereas, at low energies, g is nearly a constant. If one makes a mistake and takes $\tilde{g} = 1$ for *all* energies, one would obtain the scaling $\delta\gamma_-(z_0 - ix) \sim x^{1-2\alpha}\theta(x)$, which produces the incorrect NIBA result $F_t^{0,b} \sim 1/(\Omega t)^{2-2\alpha}$. On the other hand, if one neglects the renormalization of g and takes it as a constant, one obtains the correct scaling for long times but the scaling of the Z -factors at high energies will change with the consequence of an incorrect

description of the dynamics at short times. Therefore, for a proper description of the time dynamics on *all* time scales, it is crucial to take the vertex renormalization into account.

Summary and outlook. We have shown that the long-time behavior of open quantum systems involves logarithmic corrections which are generically quite different from those at short and intermediate times. For weak-coupling problems we propose a perturbative RG method in Liouville space with a complex flow parameter where these corrections can be calculated from a systematic expansion around the fixed points. We applied the theory to the Ohmic spin boson model and found indeed a different long-time behavior than previously predicted. Moreover, since certain terms of the RG flow are only cut off by the scale $1/t$ of the inverse time but not by decay rates Γ : it is not guaranteed for all models that the RG flow stays in the weak-coupling regime close to the fixed points irrespective the size of Γ . A prominent example of such behavior is the antiferromagnetic nonequilibrium Kondo model at bias voltage V much larger than the Kondo temperature T_K , which has been shown to be a weak-coupling problem for the determination of stationary quantities.¹⁴⁻¹⁷ However, by using the E-RTRG method of Ref. 11, it turns out that the renormalized vertices become strong for E close to the fixed points $z_n = nV - i\Gamma$, although Γ is much larger than the Kondo temperature T_K , whereas they stay small for E close to zero (which sets the point to determine stationary quantities). As a consequence, the long-time behavior can not be calculated from a weak-coupling analysis but a strong-coupling analysis is needed, which goes beyond the perturbative RG method presented in this work and poses new interesting problems for the future. Other interesting situations arise when one of the poles z_k^p is equal to zero and is a branching pole, i.e., when there is no exponential decay but a nontrivial preexponential function with logarithmic scaling. This happens typically for problems with quantum critical behavior or for reservoirs with a nonanalytic spectral density in the limit $D \rightarrow \infty$, e.g., for multichannel Kondo or sub-Ohmic spin boson models. Such systems are of particular interest since the long-time behavior is no longer suppressed by the exponential decay and the logarithmic scaling behavior becomes more visible.

This work was supported by the DFG via FOR 723. We thank V. Meden for valuable discussions.

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- ¹⁹ At finite temperature T it turns out that the branch cuts are replaced by an infinite series of poles separated by $2\pi T$.
- ²⁰ See supplementary material for the description of the general E-RTRG method and its application to the Ohmic spin-boson model.
- ²¹ We note that the pole $z_{st}^p = 0$ for $k = st$ is usually not a branching pole but an isolated pole and fulfills $P_{st}(E)Z'(E) = P_{st}(E) = |x_{st}(E)\rangle\text{Tr}$.²⁰
- ²² For special resonant conditions, the time regime $1/|z_n| \ll t \ll 1/|z_n - z_{k,k'}^p|$ is possible, where x dominates in the resolvents and different scaling can be obtained for intermediate times, see Ref.⁹ for the nonequilibrium Kondo model.
- ²³ D.P. DiVincenzo and D. Loss, Phys. Rev. B **71**, 035318 (2005).
- ²⁴ A. A. Slutskin, K.N. Bratus', A. Bergvall and V.S. Shumeiko, Europhys. Lett. **96**, 40003 (2011).
- ²⁵ We note that the exponential part of the branch cut contributions can only be obtained from self-consistent Born approximation. Within RTRG, it can be shown generically that exponential decay is expected in all orders of perturbation theory if the symmetric part of the reservoir contraction is an analytic function.⁹
- ²⁶ The exponential part has been obtained by improved NIBA calculation for α close to $1/2$, see R. Egger, H. Grabert, and U. Weiss, Phys. Rev. E **55**, R3809 (1997).
- ²⁷ O. Kashuba and H. Schoeller, unpublished.

Supplementary Material

We here present technical details of the general derivation of the RG equations and the application to the ohmic spin boson model. We start with a short summary of the E-flow scheme of real-time renormalization group (called E-RTRG henceforth).

I. THE E-RTRG METHOD

In the supplementary material of Ref. [11] it has been shown how to derive RG equations for the effective Liouvillian $L(E)$ by using the Laplace variable E as flow parameter. For quantum dots in the spin/orbital fluctuation regime, which are coupled to fermionic reservoirs with a flat d.o.s., like e.g. the Kondo model, the following RG equations have been derived in lowest order

$$\frac{\partial^2}{\partial E^2} L(E) = \frac{1}{2} \text{Diagram} = \frac{1}{2} \int d\bar{\omega}_1 \int d\bar{\omega}_2 f'(\bar{\omega}_1) f'(\bar{\omega}_2) G_{12}(E) \Pi(E_{12} + \bar{\omega}_{12}) G_{\bar{2}\bar{1}}(E_{12}) \quad (20)$$

$$\frac{\partial}{\partial E} G_{12}(E) = - \text{Diagram} - (1 \leftrightarrow 2) = - \int d\bar{\omega}_3 f'(\bar{\omega}_3) G_{13}(E) \Pi(E_{13} + \bar{\omega}_3) G_{\bar{3}\bar{2}}(E_{13}) - (1 \leftrightarrow 2) \quad (21)$$

Here, the filled double-circle represents the effective vertex $G_{12}(E)$ at zero frequencies $\omega_1 = \omega_2 = 0$ which is defined by all connected diagrams with two free reservoir lines. The index $1 = \eta\alpha\sigma$ with $\eta = \pm$ characterizes the reservoir field operators $a_{+\alpha\sigma} = a_{\alpha\sigma}^\dagger, a_{-\alpha\sigma} = a_{\alpha\sigma}$, where α is the reservoir and σ the spin index. The vertex has already been averaged over the Keldysh indices and its logarithmic frequency dependence has been neglected in leading order. The black horizontal lines between the vertices represent the full effective propagator of the local system $\Pi_{1\dots n} = \Pi(E_{1\dots n} + \bar{\omega}_{1\dots n})$ with $\Pi(E) = \frac{1}{E - L(E)}$. The indices of the energy argument refer to the green reservoir contractions running over this propagator, where, for each contraction, the index has to be taken from the left vertex which is connected to this contraction. This are precisely the same indices which have to be used to determine the energy argument for the vertex right to this propagator. We have defined $E_{1\dots n} = E + \bar{\mu}_{1\dots n} = E + \bar{\mu}_1 + \dots \bar{\mu}_n$, and $\bar{\omega}_{1\dots n} = \bar{\omega}_1 + \dots \bar{\omega}_n$, with $\bar{\mu}_i = \eta_i \mu_{\alpha_i}$ and $\bar{\omega}_i = \eta_i \omega_i$. Here, μ_α is the chemical potential of reservoir α and ω is the single-particle energy of the reservoir state relative to the chemical potential. The green lines connecting the vertices represent the reservoir contractions. Including the Keldysh indices they are given by $\gamma_{11'}^{pp'}(\bar{\omega}, \bar{\omega}') = \delta_{1\bar{1}'} \delta(\bar{\omega} + \bar{\omega}') \gamma^{p'}(\bar{\omega})$, where $\gamma^{p'}(\bar{\omega}) = p' f(p'\bar{\omega}) = f(\bar{\omega}) - \frac{1}{2} + \frac{p'}{2}$, $1 = \eta\alpha\sigma$, $\bar{1}' = -\eta'\alpha'\sigma'$, and $f(\omega) = \frac{1}{e^{\omega/T} + 1}$ denotes the Fermi function at temperature T . Since the cross at each line denotes the frequency derivative only $\frac{d}{d\bar{\omega}} \gamma^{p'}(\bar{\omega}) = f'(\bar{\omega})$ is needed and there is no explicit dependence on the Keldysh indices. This is the reason why only the vertices averaged over the Keldysh indices appear in the RG equations. However, for the perturbative determination of the initial condition it is needed to take the symmetric part $\frac{p'}{2}$ of the reservoir contraction into account. Symmetry factors $\frac{1}{n!}$ arising from the diagrammatic rules (when two vertices are connected by n equivalent lines) are explicitly quoted in (20). Finally, to calculate the frequency integrals at finite temperature, the following approximate form of the propagator has been proposed

$$\Pi(E + \bar{\omega}) \approx \frac{1}{\bar{\omega} + \chi(E)} Z(E) \quad , \quad (22)$$

where $Z(E) = \frac{1}{1 - \frac{\partial}{\partial E} L(E)}$ is the Z -factor superoperator and $\chi(E) = Z(E)(E - L(E))$. At zero temperature, this approximation is not needed since $f'(\bar{\omega}) = -\delta(\bar{\omega})$ for $T = 0$.

Obviously, the frequency integrals are well-defined in the wide band limit $D \rightarrow \infty$, i.e. we have obtained universal RG equations. This is the reason why two energy derivatives are needed for the RG of the Liouvillian, which is typical for problems with spin/orbital fluctuations. If the same formalism is applied to quantum dots in the charge fluctuation regime, where the dot states are coupled via tunneling vertices to reservoirs with a flat d.o.s., a single derivative is sufficient for convergence and one obtains in leading order the RG equation

$$\frac{\partial}{\partial E} L(E) = - \text{Diagram} = - \int d\bar{\omega}_1 f'(\bar{\omega}_1) G_1(E) \Pi(E_1 + \bar{\omega}_1) G_{\bar{1}}(E_1) \quad (23)$$

Here, in leading order, the unrenormalized vertices can be taken.

For the ohmic spin boson model, where a local 2-level system is coupled linearly to an ohmic bosonic bath, the situation is different since the reservoir contractions are dressed by the spectral density of the couplings, which is

linear in frequency. This means that the frequency integral in (23) becomes logarithmically divergent and a second derivative w.r.t. the Laplace variable is needed. To show how the reservoir contractions have to be determined we consider a bosonic bath $H_{res} = \sum_q \omega_q a_q^\dagger a_q$ with $\omega_q > 0$ and a linear coupling of the form $V = \hat{\gamma} \sum_q g_q (a_q + a_q^\dagger)$, where $\hat{\gamma}$ is a generic local operator (i.e. acts only on the states of the local quantum system) and g_q denotes the coupling between the local system and mode q of the bath. For an ohmic bath, the couplings are characterized by the spectral density

$$J(\omega) = \pi \sum_q g_q^2 \delta(\omega - \omega_q) = 2\pi \alpha \omega \theta(\omega) \rho(\omega) \quad (24)$$

where $\rho(\omega) = \frac{D^2}{D^2 + \omega^2}$ is some high-energy cutoff function, which here is chosen as a Lorentzian for convenience. α is a dimensionless coupling constant characterizing the damping, which is assumed to be small $\alpha \ll 1$. Using the notation $a_\eta(\omega) = \sum_q a_{\eta q} \delta(\omega - \omega_q) g_q$, each reservoir contraction can be expressed via the following average w.r.t. to the canonical distribution of the bosonic bath

$$\langle a_\eta(\omega) a_{\eta'}(\omega') \rangle = \delta_{\eta, -\eta'} \delta(\omega - \omega') \eta n(\eta\omega) \frac{1}{\pi} J(\omega) \quad (25)$$

where $\langle a_{\eta q} a_{\eta' q'} \rangle = \delta_{\eta, -\eta'} \delta_{qq'} \eta n(\eta\omega_q)$ has been used, with the Bose function $n(\omega) = \frac{1}{e^{\omega/T} - 1}$. Following Ref. [8], this leads to the following contraction between the reservoir field operators in Liouville space

$$\begin{aligned} \gamma_{11'}^{pp'}(\bar{\omega}, \bar{\omega}') &= \delta_{1\bar{1}'} \delta(\bar{\omega} + \bar{\omega}') \frac{1}{\pi} J(\eta\bar{\omega}) \eta p' n(p'\bar{\omega}) \\ &= \delta_{1\bar{1}'} \delta(\bar{\omega} + \bar{\omega}') 2\alpha p' \bar{\omega} n(p'\bar{\omega}) \theta(\eta\bar{\omega}) \rho(\bar{\omega}) \end{aligned} \quad (26)$$

where the indices $1 \equiv \eta$ and $\bar{1} \equiv -\eta$ involve only the creation/annihilation index and we have used $\rho(\omega) = \rho(-\omega)$. Since the local vertex operator $\hat{\gamma}$ is independent of η , the contraction (26) can be averaged over η and η' , which gives

$$\gamma^{pp'}(\omega, \omega') \equiv \sum_{11'} \gamma_{11'}^{pp'}(\omega, \omega') = \delta(\omega + \omega') \gamma^{p'}(\omega) \quad , \quad \gamma^{p'}(\omega) = 2\alpha p' \omega n(p'\omega) \rho(\omega) \quad (27)$$

In the wide band limit $D \rightarrow \infty$ we omit $\rho(\omega)$. Using $n(-\omega) = -(1 + n(\omega))$, we can split the Bose function via $n(\omega) = -\frac{1}{2} + (n(\omega) + \frac{1}{2})$ in symmetric and antisymmetric part, and obtain for (27)

$$\gamma^{p'}(\omega) = \gamma_s^{p'}(\omega) + \gamma_a(\omega) \quad , \quad \gamma_s^{p'}(\omega) = -\alpha p' \omega \quad , \quad \gamma_a(\omega) = \alpha \omega (2n(\omega) + 1) \quad (28)$$

This gives for the derivatives of the symmetric part

$$\frac{d}{d\omega} \gamma_s^{p'}(\omega) = -\alpha p' \quad , \quad \frac{d^2}{d\omega^2} \gamma_s^{p'}(\omega) = 0 \quad . \quad (29)$$

Most importantly, the first derivative is frequency independent and gives no contribution to the RG diagrams (see below). This is the reason why only the vertices averaged over the Keldysh indices appear in the RG. For the special case of zero temperature, we get $n(\omega) = -\theta(-\omega)$, and the antisymmetric part of the contraction together with its derivatives reads

$$\gamma_a(\omega) = \alpha |\omega| \quad , \quad \frac{d}{d\omega} \gamma_a(\omega) = \alpha \text{sign}(\omega) \quad , \quad \frac{d^2}{d\omega^2} \gamma_a(\omega) = 2\alpha \delta(\omega) \quad . \quad (30)$$

Using this form of the contractions, the universal RG equations follow from the formalism of Ref. [11] in leading order as

$$\frac{\partial^2}{\partial E^2} L(E) = \text{Diagram} = \int d\omega \frac{d^2 \gamma_a}{d\omega^2} G(E) \Pi(E + \omega) G(E) \quad (31)$$

$$\frac{\partial}{\partial E} G(E) = - \text{Diagram} = - \int d\omega \frac{d\gamma_a}{d\omega} G(E) \Pi(E + \omega) G(E) \Pi(E + \omega) G(E) \quad (32)$$

Here, the vertex has no further reservoir indices (only for the case of several reservoirs and different coupling operators $\hat{\gamma}$ to the reservoirs, the reservoir index has to be retained). Furthermore, the energy argument is just the Laplace variable E since there is no chemical potential in the bosonic reservoir (we allow only for energy exchange

with the environment). The frequency integrals are convergent and we note that the derivatives of the symmetric part of the contraction do not contribute in both RG diagrams. To see this for the RG diagram (32) of the vertex renormalization, one closes the integration contour in the upper half of the complex plane and obtains zero since the derivative of the symmetric part is frequency independent and the propagator is an analytic function in the upper half. However, the symmetric part will give rise to a perturbative correction in the initial condition, see below.

Decomposition of the Liouvillian.— Due to logarithmic divergencies in the high-energy cutoff D , the Liouvillian $L(E)$ will generically depend on various logarithmic terms $\sim \alpha \ln(\frac{D}{E-z_n})$, which are cut off at high energies by D and at low energies by the singularities z_n of the propagator $\Pi(E)$. In addition, since the Liouvillian has the dimension of an energy, linear terms in the Laplace variable E can occur. In the universal regime $E \ll D$, the following decomposition is useful, which explicitly exhibits the slowly varying logarithmic parts

$$L(E) = L_\Delta(E) + E L'(E) \quad . \quad (33)$$

Here, $L_\Delta(E)$ is proportional to some physical energy scale Δ (like e.g. temperature, magnetic field, decay rates, chemical potentials, level spacing, etc.) but *not* the Laplace variable E , and $L_\Delta(E)$ and $L'(E)$ are slowly varying logarithmic functions w.r.t. E . With this form we can write the propagator as

$$\Pi_{1\dots n} \equiv \Pi(E_{1\dots n} + \bar{\omega}_{1\dots n}) = \frac{1}{E_{1\dots n} + \bar{\omega}_{1\dots n} - L(E_{1\dots n} + \bar{\omega}_{1\dots n})} \quad (34)$$

$$= \frac{1}{E_{1\dots n} + \bar{\omega}_{1\dots n} - L_\Delta(E_{1\dots n} + \bar{\omega}_{1\dots n}) - (E_{1\dots n} + \bar{\omega}_{1\dots n})L'(E_{1\dots n} + \bar{\omega}_{1\dots n})} = \frac{1}{E + \chi_{1\dots n}^\Delta} Z'_{1\dots n} \quad , \quad (35)$$

where

$$\chi_{1\dots n}^\Delta = \bar{\omega}_{1\dots n} + \bar{\mu}_{1\dots n} - \tilde{L}_{1\dots n}^\Delta \quad , \quad \tilde{L}_{1\dots n}^\Delta = Z'_{1\dots n} L_{1\dots n}^\Delta \quad , \quad Z'_{1\dots n} = \frac{1}{1 - L'_{1\dots n}} \quad , \quad Z'(E) = \frac{1}{1 - L'(E)} \quad , \quad (36)$$

together with

$$L_{1\dots n}^\Delta = L_\Delta(E_{1\dots n} + \bar{\omega}_{1\dots n}) \quad , \quad L'_{1\dots n} = L'(E_{1\dots n} + \bar{\omega}_{1\dots n}) \quad , \quad Z'_{1\dots n} = Z'(E_{1\dots n} + \bar{\omega}_{1\dots n}) \quad . \quad (37)$$

Note that $Z(E) = \frac{1}{1 - \partial_E L(E)}$ and $Z'(E) = \frac{1}{1 - L'(E)}$ are not identical since

$$\partial_E L(E) = L'(E) + \{\partial_E L_\Delta(E) + E \partial_E L'(E)\} = L'(E) + O(G^2) \quad . \quad (38)$$

Below we will show that $\partial_E L_\Delta(E) + E \partial_E L'(E) \sim O(G^2)$, i.e. $Z(E)$ and $Z'(E)$ are the same in leading order up to $O(G)$.

We introduce the following graphical notations for the lines connecting the vertices

$$\Pi_{1\dots n} \equiv \bullet \text{---} \bullet \quad , \quad Z'_{1\dots n} \equiv \bullet \text{---} \bullet \quad , \quad \chi_{1\dots n}^\Delta \Pi_{1\dots n} \equiv \bullet \text{---} \chi \text{---} \bullet \quad , \quad (39)$$

such that the relation $Z'_{1\dots n} - E \Pi_{1\dots n} = \chi_{1\dots n}^\Delta \Pi_{1\dots n}$ can be written diagrammatically as

$$\bullet \text{---} \bullet - E \bullet \text{---} \bullet = \bullet \text{---} \chi \text{---} \bullet \quad . \quad (40)$$

To find RG equations for $L_\Delta(E)$ and $L'(E)$, we consider the three cases of spin/orbital fluctuations (e.g. Kondo model, Eq. (20)), charge fluctuations (e.g. quantum dots, Eq. (23)) and energy fluctuations (e.g. spin boson model, Eq. (31)) separately. For charge fluctuations, the RG equation (23) is already of first order, i.e. the Liouvillian $L(E) \approx L_\Delta(E)$ is in leading order a logarithmic function and we get

$$\text{Charge fluctuations:} \quad \frac{\partial}{\partial E} L_\Delta(E) = - \text{diagram} \quad , \quad \frac{\partial}{\partial E} L'(E) = 0 \quad (41)$$

For spin/orbital fluctuations we write the second order differential equations (20) formally as

$$\frac{\partial^2}{\partial E^2} L(E) = \frac{1}{2} \text{diagram} + \frac{\partial}{\partial E} \left\{ \frac{1}{2} \text{diagram} \right\} + O(G^3) \quad . \quad (42)$$

Thereby, the E -derivative of the Z -factors and vertices in the second diagram on the r.h.s. gives rise to terms of $O(G^3)$ and are added for convenience. We now identify the r.h.s. of this equation with the expression

$$\frac{\partial^2}{\partial E^2} L(E) = \frac{\partial}{\partial E} L'(E) + \frac{\partial}{\partial E} \left\{ \frac{\partial}{\partial E} L_\Delta + E \frac{\partial}{\partial E} L'(E) \right\} \quad (43)$$

and find

$$\frac{\partial}{\partial E} L'(E) = \frac{1}{2} \text{ (diagram: two horizontal lines with two dots each, a green loop on top with a blue cross) } \quad (44)$$

$$\frac{\partial}{\partial E} L_\Delta + E \frac{\partial}{\partial E} L'(E) = \frac{\partial}{\partial E} L(E) - L'(E) = \frac{1}{2} \text{ (diagram: two horizontal lines with two dots each, a green loop on top with a blue cross, and a dashed line below) } + O(G^3) \quad (45)$$

The second equation shows that the difference between $\partial_E L(E)$ and $L'(E)$ is indeed of $O(G^2)$ as stated above after Eq. (38). Solving (45) for $\partial_E L_\Delta(E)$ and using (44) and (40) we find in leading order

$$\text{Spin/orbital fluctuations:} \quad \frac{\partial}{\partial E} L_\Delta(E) = \frac{1}{2} \text{ (diagram: two horizontal lines with two dots each, a green loop on top with a blue cross, and a label } \chi \text{ below) } , \quad \frac{\partial}{\partial E} L'(E) = \frac{1}{2} \text{ (diagram: two horizontal lines with two dots each, a green loop on top with a blue cross) } \quad (46)$$

As required we find that $L_\Delta(E)$ is proportional to some physical energy scale appearing in $\chi_{1\dots n}^\Delta$. Note that, at finite temperature, also the frequencies $\bar{\omega}_{1\dots n}$ appear in $\chi_{1\dots n}^\Delta$, which gives a term proportional to temperature for $L_\Delta(E)$.

The same procedure can be applied to the RG equation (31) for the case of energy fluctuations with the result

$$\text{Energy fluctuations:} \quad \frac{\partial}{\partial E} L_\Delta(E) = \text{ (diagram: two horizontal lines with two dots each, a green loop on top with two blue crosses, and a label } \chi \text{ below) } , \quad \frac{\partial}{\partial E} L'(E) = \text{ (diagram: two horizontal lines with two dots each, a green loop on top with two blue crosses) } \quad (47)$$

The RG equations (41), (46) and (47), together with the vertex renormalization (21) and (32) are the final differential equations to be solved to obtain the Liouvillian $L(E)$ in the form (33). We note that the choice of the second term in (42) seems at first sight not unique since it is of higher order. However, it is not important whether the construction of the RG equations for $L_\Delta(E)$ and $L'(E)$ is unique at a certain truncation order, but the crucial point is that all corrections to this construction can be shown to be beyond leading order.

With the quantities $L_\Delta(E)$ and $L'(E)$ the propagator $\Pi(E)$ can be expressed as

$$\Pi(E) = \frac{1}{E - L(E)} = \tilde{\Pi}(E) Z'(E) \quad , \quad \tilde{\Pi}(E) = \frac{1}{E - \tilde{L}_\Delta(E)} \quad (48)$$

with

$$\tilde{L}_\Delta(E) = Z'(E) L_\Delta(E) \quad , \quad Z'(E) = \frac{1}{E - L'(E)} \quad (49)$$

To evaluate the RG equations one needs the frequency dependence of the propagator $\Pi(E + \omega)$. This is approximated by neglecting it in leading order within the logarithmic parts \tilde{L}_Δ and Z' , i.e. we use

$$\Pi(E + \omega) \approx \frac{1}{E + \omega - \tilde{L}_\Delta(E)} Z'(E) \quad (50)$$

Finally, due to the properties $\text{Tr} L(E) = 0$ and $L(E)^c = -L(-E^*)$ (with the c -transform defined by $(L^c)_{s_1 s_2, s_1' s_2'} = (L_{s_2 s_1, s_2' s_1'})^*$) [8], we note the following properties for the quantities $\tilde{L}_\Delta(E)$ and $Z'(E)$

$$\text{Tr } \tilde{L}_\Delta(E) = 0 \quad , \quad \text{Tr } Z'(E) = 1 \quad , \quad \tilde{L}_\Delta(E)^c = -\tilde{L}_\Delta(-E^*) \quad , \quad Z'(E)^c = Z'(-E^*) \quad (51)$$

Time evolution.— Once the quantities $\tilde{L}_\Delta(E)$ and $Z'(E)$ are known, the time evolution of the local density matrix ρ_t can be discussed in a straightforward way. Choosing any initial state $\rho_{t=0}$ at $t = 0$, we get in Laplace space the solution $\rho(E) = \int_0^\infty dt e^{iEt} \rho_t = i\Pi(E)\rho_{t=0}$, and in time space via inverse Laplace transform

$$\rho_t = \frac{i}{2\pi} \int_{-\infty+i0^+}^{\infty+i0^+} dE e^{-iEt} \frac{1}{E - L(E)} \rho_{t=0} = \frac{i}{2\pi} \int_{-\infty+i0^+}^{\infty+i0^+} dE e^{-iEt} \frac{1}{E - \tilde{L}_\Delta(E)} Z'(E) \rho_{t=0} \quad , \quad (52)$$

The last form of (52) is very helpful for the evaluation of the energy integral because it explicitly exhibits the slowly varying logarithmic functions $\tilde{L}_\Delta(E)$ and $Z'(E)$. To identify the singularities of the integrand we use the spectral decomposition of the Liouvillian $\tilde{L}_\Delta(E)$ in terms of its eigenvalues $\lambda_k(E)$ and corresponding projectors $P_k(E)$

$$\tilde{L}_\Delta(E) = \sum_k \lambda_k(E) P_k(E) \quad (53)$$

Since we deal with a non-hermitian superoperator, we have to distinguish the left and right eigenvectors, which we denote in Dirac notation by $|x_k(E)\rangle$ and $\langle\bar{x}_k(E)|$

$$\tilde{L}_\Delta(E) |x_k(E)\rangle = \lambda_k(E) |x_k(E)\rangle \quad , \quad \langle\bar{x}_k(E)| \tilde{L}_\Delta(E) = \langle\bar{x}_k(E)| \lambda_k(E) \quad . \quad (54)$$

The eigenvectors fulfill the orthonormalization condition $\langle\bar{x}_k(E)|x_{k'}(E)\rangle = \delta_{kk'}$ and the projectors are given by $P_k(E) = |x_k(E)\rangle\langle\bar{x}_k(E)|$ with $\sum_k P_k(E) = 1$.

Due to the condition $\text{Tr} \tilde{L}_\Delta(E) = 0$, we obtain either $\lambda_k(E) = 0$ or $\text{Tr} |x_k(E)\rangle = 0$. Therefore, the Liouvillian has always an eigenvalue zero, which we characterize by the index $k = \text{st}$ since it corresponds to the stationary state (see below). The other eigenvalues are numerated by $k = 0, \pm 1, \pm 2, \dots$. We get

$$\text{Tr} |x_{\text{st}}(E)\rangle = \sum_s \langle ss | x_{\text{st}}(E) \rangle = 1 \quad , \quad \langle\bar{x}_{\text{st}}(E) | ss \rangle = 1 \quad (55)$$

$$\text{Tr} |x_k(E)\rangle = \sum_s \langle ss | x_k(E) \rangle = 0 \quad , \quad \text{for } k = 0, \pm 1, \pm 2, \dots \quad . \quad (56)$$

As a consequence we get $P_{\text{st}}(E) = |x_{\text{st}}(E)\rangle\text{Tr}$ and the property $\text{Tr} L_\Delta(E) = \text{Tr} L'(E) = 0$ can also be written as

$$P_{\text{st}}(E) Z'(E) = P_{\text{st}}(E) \quad , \quad P_{\text{st}}(E) L_\Delta(E) = 0 \quad . \quad (57)$$

Due to the condition $\tilde{L}(E)^c = -L(-E^*)$, the eigenvalues and projectors occur always in pairs (except for $k = 0, \text{st}$ where we define $k \equiv -k$) with

$$\lambda_{-k}(E) = -\lambda_k(-E^*) \quad , \quad P_{-k}(E) = P_k(-E^*)^c \quad . \quad (58)$$

Using the spectral representation, the time evolution can be written as

$$\rho_t = \frac{i}{2\pi} \sum_k \int_\gamma dE e^{-iEt} \frac{1}{E - \lambda_k(E)} P_k(E) Z'(E) \rho_{t=0} \quad , \quad (59)$$

where γ is an integration contour which encloses the lower half of the complex plane including the real axis. Poles are located at $E = z_k^p = \lambda_k(z_k^p)$, where $z_{\text{st}}^p = 0$ is a pole at the origin. At **zero temperature**, which we consider from now on, additional nonanalytic features occur from branch cuts since $\lambda_k(E)$, $P_k(E)$ and $Z'(E)$ depend logarithmically via terms $\sim \ln(\frac{D}{E - z_n})$ generated by the ultraviolet divergencies from the high-energy cutoff D (at finite temperature the branch cuts turn into an infinite number of discrete poles separated by $2\pi T$). From the structure of the perturbation theory it can be seen that the singularities z_n are associated with poles of the propagators $\tilde{\Pi}(E_{1\dots n})$, i.e. are located at $E_{1\dots n} = z_k^p$, where $E_{1\dots n} = E + \bar{\mu}_{1\dots n}$. Therefore, the singularities $z_n = z_k^p - \bar{\mu}_{1\dots n}$ are generically given by the poles shifted by some linear combination of the chemical potentials of the reservoirs.

Using the general expressions (52) and (59), one can discuss the qualitative form of the time evolution in different time regimes. For *short times* $t \ll 1/|z_n|$, we obtain $E \sim 1/t \gg |z_n|$, i.e. the cutoff scales z_n in the logarithmic terms are unimportant and can be neglected. Furthermore, in leading order, we can replace $E \rightarrow 1/t$ in the logarithmic parts. This means that we cut off the poor man scaling equations for $\tilde{L}_\Delta(E)$ and $Z'(E)$ at the scale $E \sim 1/t$ and obtain from (52)

$$\rho_t = \frac{i}{2\pi} \int_\gamma dE e^{-iEt} \frac{1}{E - \tilde{L}_\Delta(1/t)} Z'(1/t) \rho_{t=0} = e^{-i\tilde{L}_\Delta(1/t)t} Z'(1/t) \rho_{t=0} \quad . \quad (60)$$

Expanding the exponential one finds in leading order that the logarithmic scaling of $Z'(1/t)$ at high energies determines the short time behavior.

For *intermediate and long times* $t \gtrsim 1/|z_n|$, we have to study the contributions from the poles and branch cuts in detail. All branch cuts are chosen to point into the direction of the negative imaginary axis, i.e. are located at $z_n - ix$ with $x > 0$. This can even be enforced numerically by solving the RG equations along the two paths $E = z_n + i\Lambda \pm 0^+$ with Λ real and initially given by $\Lambda = D$. Since no singularities are surrounded by the two paths, the RG flow is analytic and can be used to determine the jump across the branch cut. This is a particular advantage of the E-RTRG method, which uses a complex flow parameter in Laplace space. Our choice for the direction of the branch cuts is very convenient since $e^{-iEt} = e^{-iz_nt} e^{-xt}$ is exponentially decaying in xt , which allows an analytical evaluation of the branch cut integrals for intermediate and long times. We start with the contributions from the branch cuts starting at a pole or branching pole at z_k^p , which we evaluate by using the form (59). For the branch cut integral we set $E = z_k^p - ix \pm 0^+$ and replace in leading order $\lambda_k(E) \rightarrow z_k^p$ and the logarithmic function $P_k(E)Z'(E)$ by its average

$\bar{P}_k(z_k^p - ix)\bar{Z}'(z_k^p - ix)$ over the branch cut, where $\bar{A}(E) = \frac{1}{2}(A(E+0^+) + A(E-0^+))$. Furthermore, in leading order, we can use $x \rightarrow 1/t$ in the logarithmic functions. This gives the contribution

$$\rho_t^{k,p} \approx e^{-iz_k^p t} \frac{1}{2\pi} \int_{0^-}^{\infty} dx e^{-xt} \left(\frac{1}{-ix+0^+} - \frac{1}{-ix-0^+} \right) \bar{P}_k(z_k^p - i/t) \bar{Z}'(z_k^p - i/t) \rho_{t=0} \quad (61)$$

Using $\frac{1}{-ix+0^+} - \frac{1}{-ix-0^+} = 2\pi\delta(x)$, we obtain

$$\rho_t^{k,p} \approx e^{-iz_k^p t} \bar{P}_k(z_k^p - i/t) \bar{Z}'(z_k^p - i/t) \rho_{t=0} \quad (62)$$

i.e., for $z_k^p = \Omega_k - i\Gamma_k$, an exponential time evolution with oscillation Ω_k and decay rate Γ_k , modulated by a logarithmic scaling function. For the special term $k = \text{st}$, where $z_{\text{st}}^p = 0$, $P_{\text{st}}(E) = |x_{\text{st}}(E)\rangle\text{Tr}$ and $P_{\text{st}}(E)Z'(E) = P_{\text{st}}(E)$, we get the following contribution to the time evolution

$$\rho_t^{\text{st},p} \approx |\overline{x_{\text{st}}}(-i/t)\rangle \xrightarrow{t \rightarrow \infty} \rho_{\text{st}} = |\overline{x_{\text{st}}}(0)\rangle \quad (63)$$

i.e. we see that for $t \rightarrow \infty$ one always gets the stationary distribution ρ_{st} but, if z_{st}^p is a branching pole, logarithmic corrections can occur for the time evolution which do not decay exponentially. We note that for the models discussed here, there is no logarithmic term in the diagrammatic series involving the pole z_{st}^p . The reason is that the projector P_{st} gives always a regular contribution, provided that the symmetric part $\gamma_s^p(\omega)$ of the contraction (28) is an analytic function [8]. Providing there is no accidental pole $z_{k \neq \text{st}}^p = 0$, the pole at $E = 0$ is isolated and has no attached branch cuts.

The evaluation of a branch cut starting at a branching point z_n which is not a pole is more subtle since both $\lambda_k(E)$ and $P_k(E)Z'(E)$ can be discontinuous and cancellations can occur between the two contributions. Therefore, it is more convenient to start from the first expression of (52) involving the propagator $\Pi(E)$. Denoting by $\delta A = A_+ - A_-$ the jump across the branch and by $\bar{A} = \frac{1}{2}(A_+ + A_-)$ the average value, with $A_{\pm} = A(E \pm 0^+) = \bar{A} \pm \frac{1}{2}\delta A$, one finds for the jump of the propagator expanding in small $\delta L \sim GL$

$$\delta\Pi(E) = \Pi_+ \delta L \Pi_- = \frac{1}{E - \bar{L}} \delta L \frac{1}{E - \bar{L}} + O(\delta L^3) \quad (64)$$

Using $\overline{AB} - \bar{A}\bar{B} = \frac{1}{4}\delta A\delta B$, we get

$$\frac{1}{E - \bar{L}} = \frac{1}{E - L} + O(\delta L^2) = \sum_k \frac{1}{E - \lambda_k} P_k Z' + O(\delta L^2) = \sum_k \frac{1}{E - \bar{\lambda}_k} \bar{P}_k \bar{Z}' + O(\delta L^2) \quad (65)$$

Inserting this in (64), neglecting $O(\delta L^3)$, and approximating $E = z_n - ix \rightarrow z_n - i/t$ in the logarithmic functions $\bar{\lambda}_k$, \bar{P}_k and \bar{Z}' , we get the following result for the branch cut integral

$$\rho_t^{n,b} \approx e^{-iz_n t} \frac{1}{2\pi} \sum_{kk' \neq n} \int_0^{\infty} dx e^{-xt} \frac{1}{z_n - ix - \bar{\lambda}_k^n} \bar{P}_k^n \bar{Z}'^n \delta L(z_n - ix) \frac{1}{z_n - ix - \bar{\lambda}_{k'}^n} \bar{P}_{k'}^n \bar{Z}'^n \rho_{t=0} \quad (66)$$

where $\bar{\lambda}_k^n = \bar{\lambda}_k(z_n - i/t)$, $\bar{P}_k^n = \bar{P}_k(z_n - i/t)$ and $\bar{Z}'^n = \bar{Z}'(z_n - i/t)$. We have omitted the cases $k = n$ or $k' = n$ since we consider a branching point and not a branching pole. Since $\bar{\lambda}_k^n \sim O(z_k^p)$, we can neglect x in the denominators of the resolvents for times $t \sim 1/x \gg 1/|z_n - z_{k,k'}^p|$. For special resonant cases, where z_n comes close to z_k^p or $z_{k'}^p$, one can also define time regimes $1/|z_n| \lesssim t \ll 1/|z_n - z_{k,k'}^p|$, where x dominates in the denominators for certain values of k or k' . In any case, to evaluate the integral over x , it is necessary to know the jump of the Liouvillian $\delta L(z_n - ix)$, for which we will derive RG equations in the following.

RG equation for δL .— In leading order, the jump δL of the Liouvillian at a branch cut with $E = z_n - ix \pm 0^+$ is generated by some propagator $\Pi_{1\dots n}$ in the perturbative expansion, which is resonant, i.e. the jump of this propagator across the branch cut becomes a δ -function. To tune the branch cut of each propagator w.r.t. E along the direction of the negative imaginary axis, we first close all integration contours over the real frequencies $\bar{\omega}$ in the upper half of the complex plane, where the only nonanalytical properties are those of the Fermi/Bose-functions on the positive imaginary axis. This turns all frequency integrations to ones along the positive imaginary axis $\int d\bar{\omega} \rightarrow i \int_0^{\infty} d\bar{\omega}$ and the sign-functions of the antisymmetric part of the Fermi/Bose functions have to be replaced by their jump on the imaginary axis $\text{sign}(\bar{\omega}) \rightarrow 2$. A particular resolvent containing the eigenvalue $\lambda_k(E_{1\dots n} + i\bar{\omega}_{1\dots n})$ will then become resonant if the condition $z_n = z_k^p - \bar{\mu}_{1\dots n}$ is fulfilled. With $E_{1\dots n} = z_n + \bar{\mu}_{1\dots n} - ix \pm 0^+ = z_k^p - ix \pm 0^+$, we replace

approximately $\lambda_k \rightarrow z_k^p$, $P_k \rightarrow \bar{P}_k(z_k^p - ix)$ and $Z' \rightarrow \bar{Z}'(z_k^p - ix)$, which gives for the jump of the propagator the following δ -function

$$\begin{aligned} \delta\Pi_{1\dots n} &= \left(\frac{1}{-ix + i\bar{\omega}_{1\dots n} + 0^+} - \frac{1}{-ix + i\bar{\omega}_{1\dots n} - 0^+} \right) \bar{P}_k(z_k^p - ix) \bar{Z}'(z_k^p - ix) \\ &= 2\pi\delta(\bar{\omega}_{1\dots n} - x) \bar{P}_k(z_k^p - ix) \bar{Z}'(z_k^p - ix) \end{aligned} \quad (67)$$

Since $\bar{\omega}_{1\dots n} > 0$, the frequency integrals give only a contribution for $x > 0$. Diagrammatically, we indicate the jump of the propagator by

$$\delta\Pi_{1\dots n} \equiv \text{---} \bullet \text{---} \bigcirc \delta \text{---} \bullet \text{---} \quad . \quad (68)$$

The RG equations $\frac{\partial \delta L}{\partial E}(z_n - ix)$ can be obtained by a similar technique as the RG equations for $\frac{\partial}{\partial E}L(E)$ and $\frac{\partial^2}{\partial E^2}L(E)$. We obtain in leading order

$$\text{Charge fluctuations:} \quad \delta L(E) = \text{---} \bullet \text{---} \bigcirc \delta \text{---} \bullet \text{---} \quad (69)$$

$$\text{Spin/orbital fluctuations:} \quad \frac{\partial}{\partial E}\delta L(E) = -\frac{1}{2} \text{---} \bullet \text{---} \bigcirc \delta \text{---} \bullet \text{---} \quad (70)$$

$$\text{Energy fluctuations:} \quad \frac{\partial}{\partial E}\delta L(E) = - \text{---} \bullet \text{---} \bigcirc \delta \text{---} \bullet \text{---} \quad (71)$$

We have chosen the number of derivatives by the criterion that the frequency integrals on the r.h.s. are convergent.

These equations are nontrivial only on the branches, where $E = z_n - ix$ and $z_n = z_k^p - \bar{\mu}_{1\dots n}$. Setting this energy argument in the propagators between the vertices, we explicitly obtain together with (67) (note that we consider zero temperature)

Charge fluctuations:

$$\delta L(z_n - ix) = -2\pi i \theta(x) \bar{G}_1(z_k^p - \bar{\mu}_1 - ix) \bar{P}_k(z_k^p - ix) \bar{Z}'(z_k^p - ix) \bar{G}_1(z_k^p - ix) \quad (72)$$

Spin/orbital fluctuations:

$$\frac{\partial}{\partial x}\delta L(z_n - ix) = -2\pi \theta(x) \bar{G}_{12}(z_k^p - \bar{\mu}_{12} - ix) \bar{P}_k(z_k^p - ix) \bar{Z}'(z_k^p - ix) \bar{G}_{21}(z_k^p - ix) \quad (73)$$

Energy fluctuations:

$$\frac{\partial}{\partial x}\delta L(z_n - ix) = -4\pi\alpha \theta(x) \bar{G}(z_n - ix) \bar{P}_k(z_n - ix) \bar{Z}'(z_n - ix) \bar{G}(z_n - ix) \quad (74)$$

where $z_n = z_k^p$ in the case of energy fluctuations, and we have replaced all vertices by their average \bar{G} across the branch cut in case that they are discontinuous. The initial condition for the last two equations is $\delta L(z_n) = 0$. Up to the corrections from the logarithmic functions, we obtain $\delta L(z_n - ix) \sim \theta(x)$ for charge fluctuations and $\delta L(z_n - ix) \sim x\theta(x)$ for spin/orbital and energy fluctuations. Therefore, if x can be neglected in the resolvents of the integrand of (66), we obtain (up to logarithmic corrections) $\rho_t^{n,b} \sim 1/t$ for charge fluctuations and $\rho_t^{n,b} \sim 1/t^2$ for spin/orbital and energy fluctuations.

II. APPLICATION TO THE OHMIC SPIN BOSON MODEL

We now apply the formalism to the ohmic spin boson model at zero bias, defined by the Hamiltonian $H_{\text{tot}} = H + H_{\text{res}} + V$ with

$$H = -\frac{\Delta}{2}\sigma_x \quad , \quad H_{\text{res}} = \sum_q \omega_q a_q^\dagger a_q \quad , \quad V = \frac{1}{2}\sigma_z \sum_q g_q (a_q + a_q^\dagger) \quad . \quad (75)$$

According to (24), we use an ohmic spectral density and the reservoir contraction in Liouville space is given at zero temperature by (28)

$$\gamma^{p'}(\omega) = \gamma_s^{p'}(\omega) + \gamma_a(\omega) \quad , \quad \gamma_s^{p'}(\omega) = -\alpha p' \omega \quad , \quad \gamma_a(\omega) = \alpha |\omega| \quad . \quad (76)$$

To set up the algebra for the Liouvillian $L(E)$ and the vertices $G(E)$, we denote the two spin states of the local system by \pm and order the four states in Liouville space as $++$, $--$, $+-$, $-+$. This means that states in Liouville space

are vectors with 4 elements, corresponding to operators in usual Hilbert space. Superoperators acting in Liouville space are 4×4 -matrices. To parametrize an arbitrary 4×4 -matrix we decompose it in four 2×2 -blocks, each of which can be decomposed in the basis of the unity matrix $\sigma_0 \equiv \mathbb{1}_2$ and the three Pauli matrices σ_i ($i = 1, 2, 3 \equiv x, y, z$). For an arbitrary 4×4 -matrix A , we introduce the following elegant tensor notation

$$A = \sum_{i=0,1,2,3} \begin{pmatrix} A_{11}^i \sigma_i & A_{12}^i \sigma_i \\ A_{21}^i \sigma_i & A_{22}^i \sigma_i \end{pmatrix} = \sum_{i=0,1,2,3} \begin{pmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{pmatrix} \otimes \sigma_i = \sum_{i=0,1,2,3} A_i \otimes \sigma_i \quad , \quad (77)$$

where A_i is a 2×2 -matrix for all $i = 0, 1, 2, 3$. This notation has the advantage that a product of two 4×4 -matrices A and B can be written as

$$AB = \sum_{ij} (A_i \otimes \sigma_i) (B_j \otimes \sigma_j) = \sum_{ij} (A_i B_j) \otimes (\sigma_i \sigma_j) = (\underline{A} \cdot \underline{B}) \otimes \mathbb{1}_2 + i (\underline{A} \wedge \underline{B}) \otimes \underline{\sigma} \quad , \quad (78)$$

with $\underline{A}^T = (A_x, A_y, A_z)$, $\underline{B}^T = (B_x, B_y, B_z)$ and $\underline{\sigma}^T = (\sigma_x, \sigma_y, \sigma_z)$. The inverse of a matrix is given by

$$A^{-1} = \sum_{i=0,1,2,3} A_i^{-1} \otimes \sigma_i \quad . \quad (79)$$

Using this notation we get from the Hamiltonian the following matrix structure for the bare Liouvillian and the bare vertices

$$L^{(0)} = [H, \cdot] = \Delta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \tau_- \quad , \quad (80)$$

$$G^{(0)} = \sum_p G^{pp,(0)} = \frac{1}{2} [\sigma_z, \cdot] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \sigma_z \quad , \quad (81)$$

$$\tilde{G}^{(0)} = \sum_p p G^{pp,(0)} = \frac{1}{2} \{\sigma_z, \cdot\} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \sigma_z \quad , \quad (82)$$

where p is the Keldysh index, $[\cdot, \cdot]$ denotes the commutator and $\{\cdot, \cdot\}$ is the anti-commutator. Instead of $\mathbb{1}_2$ and σ_x , we use the matrices τ_{\pm} defined by

$$\tau_{\pm} = \frac{1}{2}(1 \pm \sigma_x) \quad , \quad \text{with} \quad \tau_{\sigma} \tau_{\sigma'} = \delta_{\sigma\sigma'} \tau_{\sigma} \quad , \quad \tau_{\sigma}^{-1} = \tau_{\sigma} \quad , \quad \sigma_z \tau_{\sigma} = \tau_{-\sigma} \sigma_z \quad . \quad (83)$$

Using (78-83), we get for the bare propagator

$$\Pi^{(0)}(E) = \frac{1}{E - L^{(0)}} = \sum_{\sigma} \Pi^{(0),\sigma} \otimes \tau_{\sigma} \quad , \quad \Pi^{(0),+} = \frac{1}{E} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \Pi^{(0),-} = \frac{1}{2} \sum_{\sigma=\pm} \frac{1}{E - \sigma \Delta} \begin{pmatrix} 1 & \sigma \\ \sigma & 1 \end{pmatrix} \quad , \quad (84)$$

and for the sequence of two vertices

$$G^{(0)} \Pi^{(0)}(E + \omega) G^{(0)} = \frac{1}{2} \sum_{\sigma=\pm} \frac{1}{E + \omega - \sigma \Delta} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \tau_{+} + \frac{1}{E + \omega} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \tau_{-} \quad , \quad (85)$$

$$G^{(0)} \Pi^{(0)}(E + \omega) \tilde{G}^{(0)} = \frac{1}{2} \sum_{\sigma=\pm} \frac{\sigma}{E + \omega - \sigma \Delta} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \tau_{+} \quad . \quad (86)$$

The vertex $\tilde{G}^{(0)}$ can only occur if the symmetric part $\gamma_s^{p'}(\omega) = -\alpha p' \omega$ of some contraction connects this vertex with a vertex $G^{(0)}$ standing left to it. If no other vertex stands between these two vertices, we see from (86) that the propagator gives a contribution $\sim \frac{1}{\omega^2}$ for large ω , i.e. together with the linear frequency dependence of $\gamma_s^{p'}(\omega)$ the frequency integration $\int d\omega$ involves an integrand $\sim \frac{1}{\omega}$ for large ω . This gives the following contribution to the effective

Liouvillian for $E \equiv E + i0^+$ slightly above the real axis

$$\begin{aligned}
L^s &= \text{Diagram: a horizontal line with two dots at the ends, a green arc above it labeled 's'} = -\alpha \int d\omega \omega G^{(0)} \Pi^{(0)}(E + \omega + i0^+) \tilde{G}^{(0)} \\
&= -\frac{1}{2}\alpha \sum_{\sigma=\pm} \int d\omega \frac{\sigma\omega}{E + \omega - \sigma\Delta + i0^+} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \tau_+ \\
&= -\frac{1}{2}\alpha \sum_{\sigma=\pm} \int d\omega \frac{-\sigma E + \Delta}{E + \omega - \sigma\Delta + i0^+} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \tau_+ \\
&= -\frac{1}{2}\alpha \sum_{\sigma=\pm} (-\sigma E + \Delta)(-i\pi) \int d\omega \delta(E + \omega - \sigma\Delta) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \tau_+ \\
&= i\pi\alpha\Delta \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \tau_+ .
\end{aligned} \tag{87}$$

We note that the frequency integral is not logarithmically divergent and can be directly calculated for $D \rightarrow \infty$, whereas the sequence (85) together with the antisymmetric part of the contraction leads to a logarithmically divergent integral which has to be treated by RG. The term L^s gives rise to a perturbative and energy independent contribution to the Liouvillian. It is not possible that the symmetric contraction $\gamma_s^{p'}(\omega)$ crosses over more than one propagator, since otherwise the integrand will be $\sim \frac{1}{\omega^2}$ for large ω and the integration contour can be closed in the upper half and gives zero since all propagators and the symmetric contraction are analytic in the upper half. Furthermore, since $L^s L^{(0)} = L^s G^{(0)} = 0$, the part L^s can not appear in any diagram involving more than two vertices. Therefore, in the universal limit $D \rightarrow \infty$, we can split the Liouvillian exactly as

$$L(E) = \hat{L}(E) + L^s \quad \text{with} \quad \hat{L}(E) = L_\Delta(E) + E L'(E) \quad , \quad L^s = i\pi\alpha\Delta \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \tau_+ \quad , \tag{88}$$

where $L_\Delta(E)$ and $L'(E)$ are logarithmic functions which can be determined from a diagrammatic series involving only the vertex $G^{(0)}$ averaged over the Keldysh indices. Since these diagrams involve always an even number of vertices, we find together with $\Pi^{(0)}(E) = \sum_\sigma \Pi_\sigma^{(0)} \otimes \tau_\sigma$, $G^{(0)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \sigma_z$ and the algebra (83), the form $L_\Delta(E) = \sum_\sigma L_\Delta^\sigma(E) \otimes \tau_\sigma$ and $L'(E) = \sum_\sigma L'_\sigma(E) \otimes \tau_\sigma$, where each diagram can only contribute to the matrix elements $L_\Delta^\sigma(E)_{22}$ and $L'_\sigma(E)_{22}$. Since the bare quantities are given by $L_\Delta^{(0)} = L^{(0)}$ and $L'^{(0)} = 0$, we can parametrize $L_\Delta(E)$ and $L'(E)$ in the form

$$L_\Delta(E) = \begin{pmatrix} 0 & 0 \\ 0 & -i\Gamma_\Delta^+(E) \end{pmatrix} \otimes \tau_+ + \begin{pmatrix} 0 & \Delta \\ \Delta & -i\Gamma_\Delta^-(E) \end{pmatrix} \otimes \tau_- \quad , \tag{89}$$

$$L'(E) = \sum_\sigma \begin{pmatrix} 0 & 0 \\ 0 & -i\Gamma'_\sigma(E) \end{pmatrix} \otimes \tau_\sigma \quad . \tag{90}$$

This gives for $\tilde{L}_\Delta(E) = Z'(E)L_\Delta(E)$ and $Z'(E) = \frac{1}{1-L'(E)}$ the parametrization

$$\tilde{L}_\Delta(E) = \begin{pmatrix} 0 & 0 \\ 0 & -i\Gamma_+(E) \end{pmatrix} \otimes \tau_+ + \begin{pmatrix} 0 & \Delta \\ Z_-(E)\Delta & -i\Gamma_-(E) \end{pmatrix} \otimes \tau_- \quad , \tag{91}$$

$$Z'(E) = \sum_\sigma \begin{pmatrix} 1 & 0 \\ 0 & Z_\sigma(E) \end{pmatrix} \otimes \tau_\sigma \quad , \tag{92}$$

with $Z_\sigma(E) = \frac{1}{1+i\Gamma'_\sigma(E)}$ and $\Gamma_\sigma(E) = Z_\sigma(E)\Gamma_\Delta^\sigma(E)$. Due to $L^s \hat{L}(E) = 0$, we can write for the full propagator

$$\Pi(E) = \frac{1}{E - L(E)} = \hat{\Pi}(E) \left(1 + L^s \frac{1}{E} \right) \quad \text{with} \quad \hat{\Pi}(E) = \frac{1}{E - \hat{L}(E)} = \tilde{\Pi}(E) Z'(E) \quad , \tag{93}$$

with $\tilde{\Pi}(E) = \frac{1}{E - \tilde{L}_\Delta(E)}$. In the RG equations, only the part $\hat{\Pi}(E)$ of the propagator contributes.

In contrast to the Liouvillian, any diagram for the effective vertex $G(E)$ will involve an odd number of bare vertices, which leads to the general form $G(E) = \sum_{i=y,z} g_i(E) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \sigma_i$ involving the sector of the Pauli matrices $\sigma_{y,z}$.

However, by inspecting the sum of mirrored diagrams, it turns out that only the sector σ_z remains. To see this consider a diagram of the form

$$G^{(0)} \Pi_{X_1}^{(0),\sigma} G^{(0)} \Pi_{X_2}^{(0),-\sigma} \dots \Pi_{X_{2n-1}}^{(0),\sigma} G^{(0)} \Pi_{X_{2n}}^{(0),-\sigma} G^{(0)} = A \otimes (\sigma_z \tau_\sigma \sigma_z \tau_{-\sigma})^n \sigma_z = A \otimes (\tau_{-\sigma} \sigma_z) \quad , \quad (94)$$

where X_i is the set of frequency indices crossing over the i -th resolvent from the left, and $\Pi_{X_i}^{(0),\sigma} = \Pi^{(0),\sigma}(E + \bar{\omega}_{X_i})$. Adding the mirrored diagram

$$G^{(0)} \Pi_{X_{2n}}^{(0),-\sigma} G^{(0)} \Pi_{X_{2n-1}}^{(0),\sigma} \dots \Pi_{X_2}^{(0),-\sigma} G^{(0)} \Pi_{X_1}^{(0),\sigma} G^{(0)} = A \otimes (\sigma_z \tau_{-\sigma} \sigma_z \tau_\sigma)^n \sigma_z = A \otimes (\tau_\sigma \sigma_z) \quad , \quad (95)$$

we get in total $A \otimes \sigma_z$. Therefore, the effective vertex is given by the parametrization

$$G(E) = g(E) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \sigma_z \quad . \quad (96)$$

From (91) one can find the four eigenvalues $\lambda_k(E)$ of $\tilde{L}_\Delta(E)$ together with the projectors $P_k(E)$. A straightforward algebra gives the result (we omit the energy argument E in all expressions)

$$\lambda_{\text{st}} = 0 \quad , \quad P_{\text{st}} Z' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \tau_+ \quad , \quad (97)$$

$$\lambda_0 = -i\Gamma_+ \quad , \quad P_0 Z' = \begin{pmatrix} 0 & 0 \\ 0 & Z_+ \end{pmatrix} \otimes \tau_+ \quad , \quad (98)$$

$$\lambda_\sigma = -i\frac{\Gamma_-}{2} + \sigma\sqrt{Z_- \Delta^2 - \Gamma_-^2/4} \quad , \quad P_\sigma Z' = \frac{\sigma}{2\sqrt{Z_- \Delta^2 - \Gamma_-^2/4}} \begin{pmatrix} \lambda_\sigma + i\Gamma_- & Z_- \Delta \\ Z_- \Delta & Z_- \lambda_\sigma \end{pmatrix} \otimes \tau_- \quad . \quad (99)$$

With these quantities, the propagator $\hat{\Pi}(E + \omega)$ appearing in the RG equations can be expressed as

$$\hat{\Pi}(E + \omega) \approx \frac{1}{E + \omega - \tilde{L}_\Delta(E)} Z'(E) = \sum_k \frac{1}{E + \omega - \lambda_k(E)} P_k(E) Z'(E) \quad , \quad (100)$$

where we have used the approximation (50) neglecting the frequency dependence in all logarithmic functions in leading order.

With the parametrization (96-100) for the vertex and the propagator, we now can easily evaluate the RG equations (47) and (32) with $\Pi(E + \omega) \rightarrow \hat{\Pi}(E + \omega)$ and the form (30) for the derivatives of the antisymmetric part of the contraction at zero temperature. We get

$$\begin{aligned} \frac{\partial}{\partial E} L_\Delta(E) &= \int d\omega \frac{d^2 \gamma_a}{d\omega^2}(\omega) G(E) \frac{\omega - \tilde{L}_\Delta(E)}{E + \omega - \tilde{L}_\Delta(E)} Z'(E) G(E) = -2\alpha G(E) \frac{\tilde{L}_\Delta(E)}{E - \tilde{L}_\Delta(E)} Z'(E) G(E) \\ &= -2\alpha \sum_k \frac{\lambda_k(E)}{E - \lambda_k(E)} G(E) P_k(E) Z'(E) G(E) \quad , \end{aligned} \quad (101)$$

$$\frac{\partial}{\partial E} L'(E) = \int d\omega \frac{d^2 \gamma_a}{d\omega^2}(\omega) G(E) \frac{1}{E + \omega - \tilde{L}_\Delta(E)} Z'(E) G(E) = 2\alpha \sum_k \frac{1}{E - \lambda_k(E)} G(E) P_k(E) Z'(E) G(E) \quad , \quad (102)$$

$$\begin{aligned} \frac{\partial}{\partial E} G(E) &= - \int d\omega \frac{d\gamma_a}{d\omega}(\omega) G(E) \frac{1}{E + \omega - \tilde{L}_\Delta(E)} Z'(E) G(E) \frac{1}{E + \omega - \tilde{L}_\Delta(E)} Z'(E) G(E) \\ &= -\alpha \sum_{kk'} G(E) P_k(E) Z'(E) G(E) P_{k'}(E) Z'(E) G(E) \int d\omega \text{sign}(\omega) \frac{1}{E + \omega - \lambda_k(E)} \frac{1}{E + \omega - \lambda_{k'}(E)} \quad , \end{aligned} \quad (103)$$

and

$$\begin{aligned} \frac{\partial}{\partial E} \tilde{L}_\Delta(E) &= \frac{\partial Z'}{\partial E}(E) L_\Delta(E) + Z'(E) \frac{\partial L_\Delta}{\partial E}(E) \\ &= 2\alpha \sum_k Z'(E) G(E) P_k(E) Z'(E) G(E) \frac{\tilde{L}_\Delta(E) - \lambda_k(E)}{E - \lambda_k(E)} \quad , \end{aligned} \quad (104)$$

$$\frac{\partial}{\partial E} Z'(E) = Z'(E) \frac{\partial L'}{\partial E} Z'(E) = 2\alpha \sum_k Z'(E) G(E) P_k(E) Z'(E) G(E) Z'(E) \frac{1}{E - \lambda_k(E)} \quad . \quad (105)$$

Using (we omit the argument E everywhere)

$$Z'GP_{\text{st}}Z'G = 0 \quad , \quad Z'GP_0Z'G = Z_+Z_-g^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \tau_- \quad , \quad (106)$$

$$Z'GP_\sigma Z'G = Z_+Z_-g^2 \frac{\sigma\lambda_\sigma}{2\sqrt{Z_-\Delta^2 - \Gamma_-^2/4}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \tau_+ \quad , \quad (107)$$

$$GP_kZ'GP_{k'}Z'G = Z_+Z_-g^3 \frac{\sigma\lambda_\sigma}{2\sqrt{Z_-\Delta^2 - \Gamma_-^2/4}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes (\delta_{k\sigma}\delta_{k'0}\tau_+ + \delta_{k0}\delta_{k'\sigma}\tau_-)\sigma_z \quad , \quad (108)$$

$$\int d\omega \text{sign}(\omega) \frac{1}{E+\omega-\lambda_k} \frac{1}{E+\omega-\lambda_{k'}} = -2 \frac{1}{\lambda_k-\lambda_{k'}} \ln \frac{E-\lambda_k}{E-\lambda_{k'}} \quad , \quad (109)$$

we obtain the RG equations

$$\frac{\partial\Gamma_+}{\partial E} = i\alpha\tilde{g} \sum_\sigma \frac{\sigma\lambda_\sigma}{\sqrt{\tilde{\Delta}^2 - \Gamma_-^2/4}} \frac{\lambda_0 - \lambda_\sigma}{E - \lambda_\sigma} \quad , \quad \frac{\partial\Gamma_-}{\partial E} = 2\alpha\tilde{g} \frac{\Gamma_- - \Gamma_+}{E - \lambda_0} \quad , \quad (110)$$

$$\frac{\partial Z_+}{\partial E} = \alpha Z_+ \tilde{g} \sum_\sigma \frac{\sigma\lambda_\sigma}{\sqrt{\tilde{\Delta}^2 - \Gamma_-^2/4}} \frac{1}{E - \lambda_\sigma} \quad , \quad \frac{\partial Z_-}{\partial E} = 2\alpha Z_- \tilde{g} \frac{1}{E - \lambda_0} \quad , \quad (111)$$

$$\frac{\partial g}{\partial E} = \alpha g \tilde{g} \sum_\sigma \frac{\sigma\lambda_\sigma}{\sqrt{\tilde{\Delta}^2 - \Gamma_-^2/4}} \frac{1}{\lambda_0 - \lambda_\sigma} \ln \frac{E - \lambda_0}{E - \lambda_\sigma} \quad , \quad (112)$$

where we have defined $\tilde{g} = Z_+Z_-g^2$, and $\tilde{\Delta} = \sqrt{Z_-\Delta}$ denotes the renormalized tunneling. The RG equations simplify considerably since we can use $\tilde{\Delta} \gg \Gamma_-$, which is fulfilled during the whole RG flow since $\Gamma_- \lesssim \alpha^2\tilde{\Delta}$ (see below). Using $\lambda_\sigma \approx \sigma\tilde{\Delta}$ and $\sqrt{\tilde{\Delta}^2 - \Gamma_-^2/4} \approx \tilde{\Delta}$, we get the final equations

$$\frac{\partial\Gamma_+}{\partial E} = i\alpha\tilde{g} \sum_\sigma \frac{\lambda_0 - \lambda_\sigma}{E - \lambda_\sigma} \quad , \quad \frac{\partial\Gamma_-}{\partial E} = 2\alpha\tilde{g} \frac{\Gamma_- - \Gamma_+}{E - \lambda_0} \quad , \quad (113)$$

$$\frac{\partial Z_+}{\partial E} = \alpha Z_+ \tilde{g} \sum_\sigma \frac{1}{E - \lambda_\sigma} \quad , \quad \frac{\partial Z_-}{\partial E} = 2\alpha Z_- \tilde{g} \frac{1}{E - \lambda_0} \quad , \quad (114)$$

$$\frac{\partial g}{\partial E} = \alpha g \tilde{g} \sum_\sigma \frac{1}{\lambda_0 - \lambda_\sigma} \ln \frac{E - \lambda_0}{E - \lambda_\sigma} \quad , \quad (115)$$

$$\frac{\partial \tilde{g}}{\partial E} = \alpha \tilde{g}^2 \sum_\sigma \left(\frac{1}{E - \lambda_\sigma} + \frac{1}{E - \lambda_0} + \frac{2}{\lambda_0 - \lambda_\sigma} \ln \frac{E - \lambda_0}{E - \lambda_\sigma} \right) \quad , \quad (116)$$

together with the initial conditions at $E = iD$

$$\Gamma_\pm|_{E=iD} = 0 \quad , \quad Z_\pm|_{E=iD} = 1 \quad , \quad g|_{E=iD} = \tilde{g}|_{E=iD} = 1 \quad . \quad (117)$$

To solve the RG equations we first consider the regime of *large energies* $|E| \gg |\lambda_0|, |\lambda_\pm|$. In this regime we get

$$\frac{\partial\Gamma_+}{\partial E} \approx 2\alpha\tilde{g} \left(\frac{\Gamma_+}{E} - i \frac{Z_-\Delta^2}{E^2} \right) \quad , \quad \frac{\partial\Gamma_-}{\partial E} \approx 2\alpha\tilde{g} \frac{\Gamma_- - \Gamma_+}{E} \quad , \quad (118)$$

$$\frac{\partial Z_\pm}{\partial E} \approx 2\alpha Z_\pm \tilde{g} \frac{1}{E} \quad , \quad \frac{\partial \tilde{g}}{\partial E} \approx \frac{2}{3} \alpha \tilde{g}^2 \sum_\sigma \frac{(\lambda_0 - \lambda_\sigma)^2}{E^3} \quad , \quad (119)$$

which leads to the result

$$\Gamma_+ \approx 2i\alpha \frac{Z_-\Delta^2}{E} \quad , \quad \Gamma_- \approx 4i\alpha^2 \frac{Z_-\Delta^2}{E} \approx 2\alpha\Gamma_+ \quad , \quad (120)$$

$$Z_\pm \approx \left(\frac{-iE}{D} \right)^{2\alpha} \quad , \quad \tilde{g} \approx 1 - \frac{2}{3} \alpha \frac{Z_-\Delta^2}{E^2} \quad . \quad (121)$$

Obviously, we have resummed in this solution all powers of logarithmic divergencies $\sim (\alpha \ln(D/E))^n$. This defines the *poor man scaling regime*, which, in time space, corresponds to the *short-time regime*. Defining a low energy scale by $\Omega = \Delta(\Omega/D)^\alpha$, we can write the solution for Z_\pm as

$$Z_\pm \approx \left(\frac{\Omega}{\Delta}\right)^2 \left(\frac{-iE}{\Omega}\right)^{2\alpha} \quad \text{with} \quad \Omega = \Delta \left(\frac{\Omega}{D}\right)^\alpha = \Delta \left(\frac{\Delta}{D}\right)^{\frac{1}{1-\alpha}}, \quad (122)$$

such that we obtain a universal function in terms of the effective tunneling Ω . Since $\Gamma_- \lesssim \alpha^2 \tilde{\Delta}$, we disregard it in the following.

We next consider the regime of *intermediate energies*, where $\alpha \ln \frac{|\lambda_k|}{|E-\lambda_k|} \ll 1$ ($k = 0, \pm$). In this regime we can solve the RG equations perturbatively in α with the result

$$\Gamma_+ \approx i\alpha \sum_\sigma (\lambda_0 - \lambda_\sigma) \ln \frac{-i(E - \lambda_\sigma)}{\Omega} \quad , \quad \tilde{g} = 1 + O(\alpha) \quad , \quad g = \left(\frac{\Delta}{\Omega}\right)^2 + O(\alpha) \quad , \quad (123)$$

$$Z_+ \approx \left(\frac{\Omega}{\Delta}\right)^2 \left(1 + \alpha \sum_\sigma \ln \frac{-i(E - \lambda_\sigma)}{\Omega}\right) \quad , \quad Z_- \approx \left(\frac{\Omega}{\Delta}\right)^2 \left(1 + 2\alpha \ln \frac{-i(E - \lambda_0)}{\Omega}\right) \quad , \quad (124)$$

where all integration constants have been fixed by comparison with the solution at high energies. In this solution we have resummed all powers of logarithmic divergencies $\sim (\alpha \ln(D/\Omega))^n$, and have expanded in the other small logarithmic functions. It defines the *weak-coupling expansion regime*, which, in time space, corresponds to the *intermediate time regime*. This weak-coupling expansion is equivalent to the one developed in Ref. [17].

From the perturbative solution at intermediate energies, we can already derive the leading order result for the real and imaginary parts of the pole positions, defined by $z_k = \lambda_k(z_k)$ (note that $z_k = z_k^p$ since there is no chemical potential in the present problem), with $\lambda_0 = -i\Gamma_+$ and $\lambda_\sigma = \sigma\sqrt{Z_-}\Delta$. For intermediate energies, we can approximately set $\lambda_k(E) \approx z_k(1 + O(\alpha))$ in (123) and (124), and find directly $z_0 = -i\Gamma_+(z_0) \sim O(\alpha)$ and $z_\sigma = \sigma\sqrt{Z_-}(z_\sigma)\Delta = \sigma\Omega(1 + O(\alpha))$. Using this result in (123) and (124) to calculate the $O(\alpha)$ correction, we find

$$iz_0 = \Gamma_+(z_0) \approx i\alpha \sum_\sigma (z_0 - z_\sigma) \ln \frac{-i(z_0 - z_\sigma)}{\Omega} \approx -i\alpha\Omega \sum_\sigma \sigma \ln(i\sigma) = \pi\alpha\Omega \quad , \quad (125)$$

$$\frac{z_+}{\Delta} = \sqrt{Z_-}(z_+) \approx \frac{\Omega}{\Delta} \left(1 + \alpha \ln \frac{-i(z_+ - z_0)}{\Omega}\right) \approx \frac{\Omega}{\Delta} (1 + \alpha \ln(-i)) = \frac{\Omega}{\Delta} (1 - i\frac{\pi}{2}\alpha) \quad , \quad (126)$$

i.e. with $z_- = -z_+^*$, the pole positions are given by

$$z_0 = -i\Gamma \quad , \quad z_\pm = \pm\Omega - i\frac{\Gamma}{2} \quad \text{with} \quad \Gamma = \pi\alpha\Omega \quad . \quad (127)$$

Finally, we consider the regime of *small energies*, where the Laplace variable E approaches one of the singularities z_k , such that $\alpha \ln \frac{|E-z_k|}{\Omega} \sim O(1)$. First, we note that the RG equation (115) for the vertex function $g(E)$ leads to a very weak logarithmic correction $\sim \alpha(E - z_k) \ln(E - z_k)$ close to the singularities which can be neglected. Therefore, we take the constant value $g \approx (\Delta/\Omega)^2$ at intermediate and small energies. In contrast, the vertex function $\tilde{g} = Z_+Z_-g^2$ behaves very differently. It is approximately a constant $\tilde{g} \approx 1$ for high and intermediate energies, but has strong logarithmic corrections $\sim \alpha \ln(E - z_k)$ close to the singularities, which arise from corresponding singularities of the Z-factors Z_\pm . We start with the fixed point analysis around z_0 , where $\Gamma_+(E) \approx \Gamma$ and $Z_+(E) \approx (\Omega/\Delta)^2$ behave smoothly. In contrast, $Z_-(E)$ has a logarithmic singularity, which can be determined from the RG equation (114)

$$\frac{\partial Z_-}{\partial E} \approx 2\alpha Z_- \tilde{g} \frac{1}{E - z_0} = 2\alpha Z_-^2 Z_+ g^2 \frac{1}{E - z_0} \approx 2\alpha Z_-^2 \left(\frac{\Delta}{\Omega}\right)^2 \frac{1}{E - z_0} \quad (128)$$

$$\Rightarrow \quad \frac{1}{Z_-(E)} \approx \text{const} - 2\alpha \left(\frac{\Delta}{\Omega}\right)^2 \ln \frac{-i(E - z_0)}{\Omega} \quad . \quad (129)$$

Fixing the integration constant by comparing with the solution (124) at intermediate energies, we find for E close to z_0

$$g(E) \approx \left(\frac{\Delta}{\Omega}\right)^2 \quad , \quad \Gamma_+(E) \approx \Gamma \quad , \quad Z_+(E) \approx \left(\frac{\Omega}{\Delta}\right)^2 \quad , \quad Z_-(E) \approx \left(\frac{\Omega}{\Delta}\right)^2 \frac{1}{1 - 2\alpha \ln \frac{-i(E - z_0)}{\Omega}} \quad . \quad (130)$$

Close to the singularity z_+ , we get $Z_-(E) \approx (\Omega/\Delta)^2$ but $\Gamma_+(E)$ and $Z_+(E)$ have strong logarithmic corrections, which can be determined from the RG equations (113) and (114) in the following way

$$\begin{aligned} \frac{\partial Z_+}{\partial E} &\approx \alpha Z_+ \tilde{g} \frac{1}{E - z_+} = \alpha Z_+^2 Z_- g^2 \frac{1}{E - z_+} \approx \alpha Z_+^2 \left(\frac{\Delta}{\Omega} \right)^2 \frac{1}{E - z_+} \\ \Rightarrow \quad \frac{1}{Z_+(E)} &\approx \text{const} - \alpha \left(\frac{\Delta}{\Omega} \right)^2 \ln \frac{-i(E - z_+)}{\Omega} \stackrel{(124)}{\Rightarrow} Z_+(E) \approx \left(\frac{\Omega}{\Delta} \right)^2 \frac{1}{1 - \alpha \ln \frac{-i(E - z_+)}{\Omega}} \quad , \quad (131) \\ \frac{\partial \Gamma_+}{\partial E} &\approx i\alpha \tilde{g} \frac{\lambda_0 - z_+}{E - z_+} \approx -i\alpha Z_+ Z_- g^2 \frac{\Omega}{E - z_+} \approx -i\alpha Z_+ \left(\frac{\Delta}{\Omega} \right)^2 \frac{\Omega}{E - z_+} \stackrel{(131)}{\approx} -i\alpha \frac{1}{1 - \alpha \ln \frac{-i(E - z_+)}{\Omega}} \frac{\Omega}{E - z_+} \\ \Rightarrow \quad \Gamma_+(E) &\approx \text{const} + i\Omega \ln \left(1 - \alpha \ln \frac{-i(E - z_+)}{\Omega} \right) \stackrel{(123)}{\Rightarrow} \Gamma_+(E) \approx \frac{\Gamma}{2} + i\Omega \ln \left(1 - \alpha \ln \frac{-i(E - z_+)}{\Omega} \right) \quad , \quad (132) \end{aligned}$$

where again the integration constants have been fixed by comparison with the solutions at intermediate energies. Using a similar analysis close to z_- , we obtain for E close to z_σ the result

$$g(E) \approx \left(\frac{\Delta}{\Omega} \right)^2 \quad , \quad \Gamma_+(E) \approx \frac{\Gamma}{2} + i\sigma \Omega \ln \left(1 - \alpha \ln \frac{-i(E - z_\sigma)}{\Omega} \right) \quad , \quad (133)$$

$$Z_+(E) \approx \left(\frac{\Omega}{\Delta} \right)^2 \frac{1}{1 - \alpha \ln \frac{-i(E - z_\sigma)}{\Omega}} \quad , \quad Z_-(E) \approx \left(\frac{\Omega}{\Delta} \right)^2 \quad . \quad (134)$$

With these results we can also evaluate the RG equation (74) for the jump $\delta L(z_k - ix)$ of the Liouvillian, which we parametrize as

$$\delta L(E) = -i \sum_\sigma \delta \gamma_\sigma(E) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \tau_\sigma \quad . \quad (135)$$

Using the algebra (106) and (107) together with our results (123), (124), (130), (133) and (134) at intermediate and small energies, we get for $x \lesssim \Omega$ from (74)

$$-i \frac{\partial}{\partial x} \delta \gamma_+(z_\sigma - ix) = -4\pi\alpha\theta(x) Z_-(z_\sigma - ix) g(z_\sigma - ix)^2 \frac{1}{2} \approx -2\pi\alpha\theta(x) \left(\frac{\Delta}{\Omega} \right)^2 \quad , \quad (136)$$

$$-i \frac{\partial}{\partial x} \delta \gamma_-(z_0 - ix) = -4\pi\alpha\theta(x) Z_+(z_0 - ix) g(z_0 - ix)^2 \approx -4\pi\alpha\theta(x) \left(\frac{\Delta}{\Omega} \right)^2 \quad , \quad (137)$$

with the solution

$$\delta \gamma_+(z_\sigma - ix) \approx -2\pi i \alpha \left(\frac{\Delta}{\Omega} \right)^2 x \theta(x) \quad , \quad \delta \gamma_-(z_0 - ix) \approx -4\pi i \alpha \left(\frac{\Delta}{\Omega} \right)^2 x \theta(x) \quad . \quad (138)$$

Time evolution.— With the results for the Liouvillian we now can evaluate the time evolution. We start with the *short time regime* $\Omega t \ll 1$. Using (60), (92) and (122), we find

$$\rho_t \approx \begin{pmatrix} 1 & 0 \\ 0 & (\frac{1}{Dt})^{2\alpha} \end{pmatrix} \otimes \mathbb{1}_2 \rho_{t=0} \quad . \quad (139)$$

The *intermediate and long time regime* $\Omega t \gtrsim 1$ is based on the pole and branch cut contributions given by (62) and (66), respectively. Thereby, we have to consider that the full propagator (93) involves the correction $\hat{\Pi}(E) L^s \frac{1}{E}$. Since $L^s \sim O(\alpha)$, this leads to a negligible $O(\alpha^2)$ -correction to the branch cut contributions (66) but the pole contribution (62) changes to

$$\rho_t^{\text{st},p} = \rho_{\text{st}} = \left(P_{\text{st}}(0) Z'(0) + \hat{\Pi}(0) L^s \right) \rho_{t=0} \approx \left(P_{\text{st}}(0) Z'(0) - \sum_{k=0,\pm} P_k(0) Z'(0) L^s \frac{1}{z_k} \right) \rho_{t=0} \quad , \quad (140)$$

$$\rho_t^{k,p} = e^{-iz_k t} P_k(z_k) Z'(z_k) \left(1 + L^s \frac{1}{z_k} \right) \rho_{t=0} \quad \text{for } k = 0, \pm \quad , \quad (141)$$

where we have used $P_{\text{st}} Z' L^s = 0$, $\lambda_k(0) \approx z_k$ and the fact that all poles are isolated up to leading order truncation. Using the form (88) for L^s together with the results (97-99) for the projectors (where we neglect Γ_-), we find

$$P_{\text{st}} Z' L^s = 0 \quad , \quad P_0 Z' L^s = i\pi\alpha\Delta Z_+ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \tau_+ \quad , \quad P_\sigma Z' L^s = 0 \quad . \quad (142)$$

Inserting these results in (140) and (141), we find together with $Z_+(0) \approx Z_+(z_0) \approx Z_-(z_\sigma) \approx (\Omega/\Delta)^2$ and (127)

$$\rho_t^{\text{st},p} = \rho_{\text{st}} \approx \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{i\pi\alpha\Delta Z_+(0)}{z_0} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \otimes \tau_+ \rho_{t=0} \approx \begin{pmatrix} 1 & 0 \\ \Omega/\Delta & 0 \end{pmatrix} \otimes \tau_+ \rho_{t=0} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ \Omega/\Delta \end{pmatrix} \quad , \quad (143)$$

$$\rho_t^{0,p} = e^{-iz_0 t} Z_+(z_0) \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{i\pi\alpha\Delta}{z_0} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \otimes \tau_+ \rho_{t=0} \approx e^{-iz_0 t} \begin{pmatrix} 0 & 0 \\ -\Omega/\Delta & (\Omega/\Delta)^2 \end{pmatrix} \otimes \tau_+ \rho_{t=0} \quad , \quad (144)$$

$$\begin{aligned} \rho_t^{\sigma,p} &= e^{-iz_\sigma t} \frac{\sigma}{2\sqrt{Z_-(z_\sigma)\Delta}} \begin{pmatrix} \sigma\sqrt{Z_-(z_\sigma)\Delta} & Z_-(z_\sigma)\Delta \\ Z_-(z_\sigma)\Delta & \sigma Z_-(z_\sigma)\sqrt{Z_-(z_\sigma)\Delta} \end{pmatrix} \otimes \tau_- \rho_{t=0} \\ &\approx e^{-iz_\sigma t} \frac{1}{2} \begin{pmatrix} 1 & \sigma\Omega/\Delta \\ \sigma\Omega/\Delta & (\Omega/\Delta)^2 \end{pmatrix} \otimes \tau_- \rho_{t=0} \quad . \end{aligned} \quad (145)$$

Finally, by using the algebra of the projectors $P_k Z'$ and the jump δL according to (97-99) and (135), we can write the branch cut contribution (66) as

$$\rho_t^{0,b} = e^{-iz_0 t} \frac{1}{2\pi} \sum_{\sigma\sigma'} \int_0^\infty dx \frac{e^{-xt}}{(z_0 - ix - \bar{\lambda}_\sigma^0)(z_0 - ix - \bar{\lambda}_{\sigma'}^0)} \bar{P}_\sigma^0 \bar{Z}'^0 \delta L(z_0 - ix) \bar{P}_{\sigma'}^0 \bar{Z}'^0 \rho_{t=0} \quad , \quad (146)$$

$$\rho_t^{\sigma,b} = e^{-iz_\sigma t} \frac{1}{2\pi} \sum_{\sigma\sigma'} \int_0^\infty dx \frac{e^{-xt}}{(z_\sigma - ix - \bar{\lambda}_0^\sigma)^2} \bar{P}_0^\sigma \bar{Z}'^\sigma \delta L(z_\sigma - ix) \bar{P}_0^{\sigma'} \bar{Z}'^{\sigma'} \rho_{t=0} \quad . \quad (147)$$

For $\Omega t \gg 1$ we can use

$$z_0 - ix - \bar{\lambda}_\sigma^0 \approx -\bar{\lambda}_\sigma^0 \approx -\sigma\sqrt{\bar{Z}_-^0} \Delta \quad , \quad z_\sigma - ix - \bar{\lambda}_0^\sigma \approx z_\sigma + i\bar{\Gamma}_+^\sigma \quad , \quad (148)$$

where $\bar{Z}_-^0 = \bar{Z}_-(z_0 - i/t)$ and $\bar{\Gamma}_+^\sigma = \bar{\Gamma}_+(z_\sigma - i/t)$ can be calculated from (130) and (133) as

$$\bar{Z}_-^0 \approx \left(\frac{\Omega}{\Delta}\right)^2 \frac{1}{1 + 2\alpha \ln(\Omega t)} \quad , \quad \bar{\Gamma}_+^\sigma \approx \frac{\Gamma}{2} + i\sigma\Omega \ln(1 + \alpha \ln(\Omega t)) \quad . \quad (149)$$

Furthermore, due to the algebra of the projectors $P_k Z'$ and the jump δL , we can use

$$P_\sigma Z' \delta L P_{\sigma'} Z' = -\frac{i}{4} \delta\gamma_- Z_- \begin{pmatrix} \sigma\sigma' & \sigma\sqrt{\bar{Z}_-} \\ \sigma'\sqrt{\bar{Z}_-} & Z_- \end{pmatrix} \otimes \tau_- \quad , \quad (150)$$

$$P_0 Z' \delta L P_0 Z' = -i\delta\gamma_+ (Z_+)^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \tau_+ \quad . \quad (151)$$

Inserting (148-151) in (146) and (147), and using the result (138) for $\delta\gamma_-(z_0 - ix)$ and $\delta\gamma_+(z_\sigma - ix)$, we obtain

$$\begin{aligned} \rho_t^{0,b} &\approx e^{-iz_0 t} \left(-\frac{i}{2\pi\Delta^2}\right) \int_0^\infty dx e^{-xt} \delta\gamma_-(z_0 - ix) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \tau_- \rho_{t=0} \\ &= e^{-iz_0 t} \left(-\frac{2\alpha}{\Omega^2}\right) \int_0^\infty dx e^{-xt} x \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \tau_- \rho_{t=0} \\ &= e^{-iz_0 t} \left(-\frac{2\alpha}{(\Omega t)^2}\right) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \tau_- \rho_{t=0} \quad , \end{aligned} \quad (152)$$

$$\begin{aligned} \rho_t^{\sigma,b} &\approx e^{-iz_\sigma t} \left(-\frac{i}{2\pi}\right) \left(\frac{\bar{Z}_+^\sigma}{z_\sigma + i\bar{\Gamma}_+^\sigma}\right)^2 \int_0^\infty dx e^{-xt} \delta\gamma_+(z_\sigma - ix) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \tau_+ \rho_{t=0} \\ &= e^{-iz_\sigma t} \left(-\frac{\alpha f_t}{\Delta^2}\right) \int_0^\infty dx e^{-xt} x \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \tau_+ \rho_{t=0} \\ &= e^{-iz_\sigma t} \left(-\frac{\alpha f_t}{(\Delta t)^2}\right) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \tau_+ \rho_{t=0} \quad , \end{aligned} \quad (153)$$

where we have defined the logarithmic scaling function

$$f_t = \left(\frac{1}{(1 + \alpha \ln(\Omega t))(1 - \ln(1 + \alpha \ln(\Omega t)))} \right)^2. \quad (154)$$

Eqs. (143-145) and (152-154) are the final results for the time evolution in the regime $\Omega t \gg 1$. We note that the branch cut integrals (146) and (147) can also be calculated exactly in terms of exponential integrals, extending the applicability range to the time regime $\Omega t \gtrsim 1$.